Love-for-Variety

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Teaching Slides

Some Backgrounds

“Non-CES Aggregators: A Guided Tour”
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- We all love using CES, because it is tractable.

- CES is tractable because it has many knife-edge properties, which also make it restrictive.

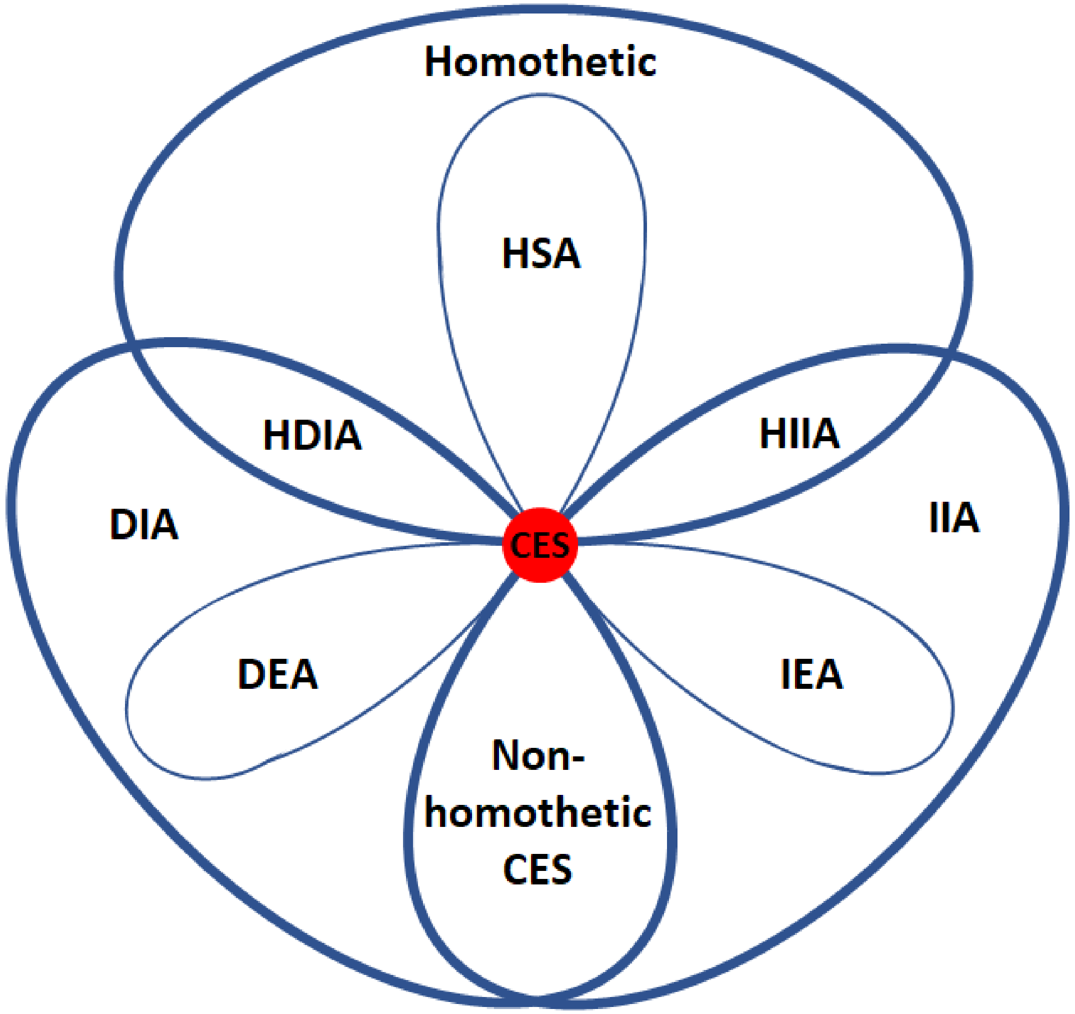
- For some purposes, we need to drop some properties.

- Many look for an alternative, such as Stone-Geary, translog, etc. But they have their own drawbacks.

- My Approach: Relax only those properties we need to drop and keep the rest to retain the tractability of CES as much as possible.

- Depending on which properties are kept, we come up with many different classes of non-CES demand systems.

- Which class should be used depends on the applications.



Introduction

Love-for-Variety (LV): Utility (productivity) gains from increasing variety of consumer goods (intermediate inputs).

- A natural consequence of the convexity of the utility (production) function.

- Following the work of Dixit-Stiglitz (1977), Krugman (1980), Ethier (1982), Romer (1987), it has become a central concept in economic growth (Grossman-Helpman 1993; Gancia-Zillibotti 2005, Acemoglu 2008), international trade (Helpman-Krugman 1995), and economic geography (Fujita-Krugman-Venables 1999).

- Commonly discussed in monopolistic competition settings, but also useful in other contexts, such as gains from trade in Armington-type competitive models of trade.

- The LV measure under CES: $\mathcal{L} = 1/(\sigma - 1) > 0,$ where $\sigma > 1$ captures 2 related but distinct concepts,
    - Elasticity of Substitution (ES) across different goods
    - Price Elasticity (PE) of demand for each good.
  - **Appealing features:**
    - LV is inversely related to ES (and PE).
    - Knowing PE tells you everything you need to know about ES and LV.
  - **Unappealing features:**
    - LV is constant. Intuitively, LV should decline as the variety increases. *In this respect, some find "Ideal variety approach" more appealing, but it is less tractable than "Love-for-variety approach."*
    - The relation btw PE, ES, & LV are hard-wired under CES, with no flexibility. To "account for" the gap btw the revealed LV and CES-implied LV, one often introduces "the Benassy residual," whose estimate depends on CES.

The Questions: What happens to LV if we move away from CES?

- How is LV related to the underlying demand structure, such as ES or PE?

ES and PE are distinct concepts outside of CES, which could play different roles shaping LV.

- How biased are our estimates of LV and of the Benassy residuals if we incorrectly assume CES?
- Under what conditions does LV decline as the variety of available goods increases?

*Does it help to introduce the empirically plausible **2nd Law of demand** (PE higher at a higher price)?*

- Can we develop "Love-for-variety approach" with diminishing LV, which is also tractable?

Our Approach:

- First, we formally define the two measures: **Substitutability,** $\mathcal{S}(V)$, & **Love-for-Variety,** $\mathcal{L}(V)$.
  - Both depend only on V (the variety of available goods) under general homothetic symmetric demand systems.
  - Under CES, $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are both constant with $\mathcal{L}(V) = \frac{1}{\sigma-1} = \frac{1}{\mathcal{S}(V)-1}$.
- What if $\mathcal{S}(V)$ varies with V?
  - One might intuitively think "**The 2nd Law of demand** $\Longrightarrow$ **Increasing** $\mathcal{S}(V) \Longrightarrow$ **Diminishing** $\mathcal{L}(V)$."
  - It turns out that this is not true in general.
- The CES formula may also over- or under-estimate LV; Both $\mathcal{L}(V) > \frac{1}{\mathcal{S}(V)-1}$ & $\mathcal{L}(V) < \frac{1}{\mathcal{S}(V)-1}$ are possible.

Almost anything goes. Homotheticity (& symmetry) too broad to impose much restrictions btw PE, $\mathcal{S}(V)$ & $\mathcal{L}(V)$.
To make further progress, we turn to FIVE classes of non-CES, each obtained as a natural departure from CES.

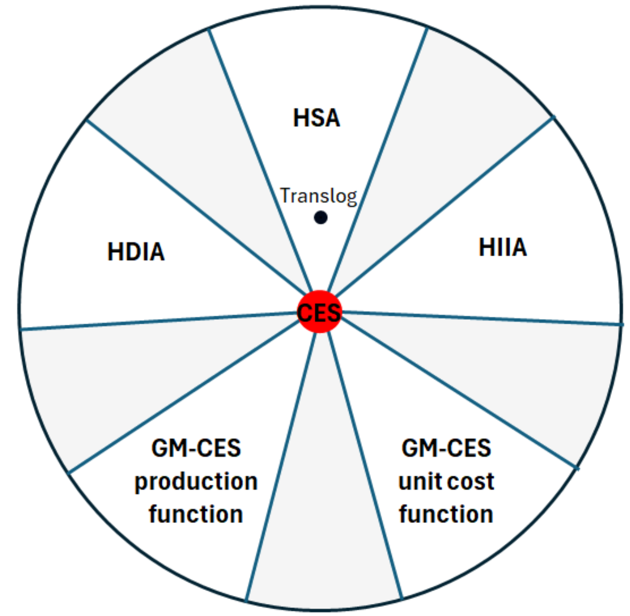
Five Classes of non-CES

They are pairwise disjoint with the sole exception of CES.

Two Classes of Geometric Means of CES (GM-CES)

Theorem 1 (GM-CES):
$\mathcal{S}(V)$ & $\mathcal{L}(V)$ both constant, with $\mathcal{L}(V) > 1/(\mathcal{S}(V) - 1)$ unless CES.

- $\mathcal{S}(V)$ only determines the lower bound of $\mathcal{L}(V)$
- The CES formula for LV underestimates $\mathcal{L}(V)$ and overestimates the Benassy residuals.



Homothetic symmetric demand systems with gross substitutes

Three Classes: H.S.A., HDIA, and HIIA.

- PE = $\zeta_\omega \equiv \zeta(p_\omega/\mathcal{A}(\mathbf{p}))$, where $\mathcal{A}(\mathbf{p})$ is linear homogeneous, a sufficient statistic for the cross-price effects.

Theorem 2: Under H.S.A., HDIA, and HIIA,

i) $\zeta'(p_\omega/\mathcal{A}(\mathbf{p})) \gtreqless 0 \Leftrightarrow \mathcal{S}'(V) \gtreqless 0$.

ii) $\mathcal{S}'(V) \gtreqless 0$ for all $V > 0 \Longrightarrow \mathcal{L}'(V) \lesseqgtr 0$ for all $V > 0$. The converse is not true.

iii) $\mathcal{L}'(V) = 0$ for all $V > 0 \Leftrightarrow \mathcal{S}'(V) = 0$ for all $V > 0$, which occurs iff CES.

Theorem 3: $\mathcal{L}'(V) \lesseqgtr 0 \Leftrightarrow \mathcal{L}(V) \lesseqgtr 1/(\mathcal{S}(V) - 1)$.

Corollary of Theorem 2 and 3: The 2nd law $\Leftrightarrow$ Increasing $\mathcal{S}(V) \Longrightarrow$ Diminishing $\mathcal{L}(V) \Leftrightarrow$ The CES formula overestimates $\mathcal{L}(V)$ and underestimates the Benassy residuals.

Theorem 4: As $V \to \infty$, $\mathcal{L}(V) - 1/(\mathcal{S}(V) - 1) \to 0$.

An Application: Gains from Trade in a Simple Armington Model of Competitive Trade

- Theorems 1-4 are about the *demand system*, independent of the supply-side, how the variety change is modelled. It could be pure discovery, innovation by the public sector, or by the private sector, which could be monopolistic, oligopolistic, or monopolistically competitive, etc.
- Nevertheless, we illustrate the implications in a simple Armington model of trade btw 2 countries, which produce different sets of goods. (See our 2020 paper for some implications in a Dixit-Stiglitz model.)

Among other things, we show:

- Under the 2 classes of GM-CES: $\ln(GT) = \mathcal{L}^{GMCES}\ln(1/\lambda) > \frac{\ln(1/\lambda)}{\varsigma^{GMCES}-1}$, $\lambda =$ the domestic expenditure share.

  The ACR formula holds with $\mathcal{L}^{GMCES}$. The CES formula underestimates GT under GM-CES.
- Under the 3 classes: λ is no longer a sufficient statistic. A smaller λ increases GT, but its implications also depend on whether it is due to a small size of the country, or a larger size of the trading partner.

E.g., With the choke price. GT is increasing in the size of the trading partner, but it is bounded, unlike CES. CES may overestimate gains from trade with a large country.

A note: Neither symmetry nor homotheticity are as restrictive as they look.

- By nesting symmetric homothetic demand systems into an upper-tier asymmetric/nonhomothetic demand system, we can create an asymmetric/nonhomothetic demand system.
- Homotheticity is indeed *an advantage*, which makes it applicable to a sector-level analysis in multi-sector settings.
- Moreover, one key message is that symmetry/homotheticity restrictions are *not restrictive enough*-- "Almost anything goes,"-- that we need to look for more restrictions to make further progress.

General Symmetric Homothetic Demand Systems

General Symmetric Homothetic (Input) Demand System

Consider demand system for a continuum of differentiated inputs generated by symmetric CRS production technology.

CRS Production Function	**Unit Cost Function**
$X(\mathbf{x}) \equiv \min_{\mathbf{p}} \left\{ \mathbf{px} = \int_\Omega p_\omega x_\omega d\omega \,\middle|\, P(\mathbf{p}) \geq 1 \right\}$	$P(\mathbf{p}) \equiv \min_{\mathbf{x}} \left\{ \mathbf{px} = \int_\Omega p_\omega x_\omega d\omega \,\middle|\, X(\mathbf{x}) \geq 1 \right\}$

$\mathbf{x} = \{x_\omega; \omega \in \bar{\Omega}\}$**:** the input quantity vector; $\mathbf{p} = \{p_\omega; \omega \in \bar{\Omega}\}$**:** the input price vector.

$\bar{\Omega}$, the continuum set of all potential inputs. $\Omega \subset \bar{\Omega}$, the set of available inputs with its mass $V \equiv |\Omega|$.

$\bar{\Omega} \backslash \Omega$: the set of unavailable inputs, $x_\omega = 0$ and $p_\omega = \infty$ for $\omega \in \bar{\Omega} \backslash \Omega$.

Inputs are *inessential*, i.e., $\bar{\Omega} \backslash \Omega \neq \emptyset$ does NOT imply $X(\mathbf{x}) = 0 \Leftrightarrow P(\mathbf{p}) = \infty$.

Duality: Either $X(\mathbf{x})$ or $P(\mathbf{p})$ can be a *primitive*, if linear homogeneity, monotonicity & strict quasi-concavity satisfied

Demand System

Demand Curve (from Shepherd's Lemma)	**Inverse Demand Curve**
$x_\omega = \frac{\partial P(\mathbf{p})}{\partial p_\omega} X(\mathbf{x})$	$p_\omega = P(\mathbf{p}) \frac{\partial X(\mathbf{x})}{\partial x_\omega}$

From Euler's Homogenous Function Theorem,

$$\mathbf{px} = \int_\Omega p_\omega x_\omega d\omega = \int_\Omega p_\omega \frac{\partial P(\mathbf{p})}{\partial p_\omega} X(\mathbf{x}) d\omega = \int_\Omega P(\mathbf{p}) \frac{\partial X(\mathbf{x})}{\partial x_\omega} x_\omega d\omega = P(\mathbf{p}) X(\mathbf{x}) = E.$$

The value of inputs is equal to the total value of output under CRS.

Budget Share of $\omega \in \Omega$:	$s_\omega \equiv \dfrac{p_\omega x_\omega}{\mathbf{px}} = \dfrac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \dfrac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} \equiv s(p_\omega, \mathbf{p}) = \dfrac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} \equiv s^*(x_\omega, \mathbf{x})$

Homogeneity of degree zero → $s_\omega = s(1, \mathbf{p}/p_\omega) = s^*(1, \mathbf{x}/x_\omega)$.
In general, it depends on the whole *distribution* of the prices (quantities) divided by its own price (quantity).

Definition: Gross Substitutability

$$\frac{\partial \ln s(p_\omega; \mathbf{p})}{\partial \ln p_\omega} < 0 \Leftrightarrow \frac{\partial \ln s^*(x_\omega;\ \mathbf{x})}{\partial \ln x_\omega} > 0$$

Price Elasticity of Demand for $\omega \in \Omega$	$\zeta_\omega \equiv -\dfrac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega; \mathbf{p}) \equiv 1 - \dfrac{\partial \ln s(p_\omega; \mathbf{p})}{\partial \ln p_\omega} = \zeta^*(x_\omega; \mathbf{x}) \equiv \left[1 - \dfrac{\partial \ln s^*(x_\omega;\ \mathbf{x})}{\partial \ln x_\omega}\right]^{-1} > 1.$

Homogeneity of degree zero implies → $\zeta_\omega = \zeta(1, \mathbf{p}/p_\omega) = \zeta^*(1, \mathbf{x}/x_\omega)$.
In general, it depends on the whole *distribution* of prices (quantities) divided by its own price (quantity).

Definition: The 2nd Law of Demand

$$\frac{\partial \ln \zeta(p_\omega; \mathbf{p})}{\partial \ln p_\omega} > 0 \Leftrightarrow \frac{\partial \ln \zeta^*(x_\omega; \mathbf{x})}{\partial \ln x_\omega} < 0.$$

Clearly, CES does not satisfy the 2nd Law.

Substitutability Measure Across Different Goods

Unit Quantity Vector: $\mathbf{1}_\Omega \equiv \{(1_\Omega)_\omega; \omega \in \bar{\Omega}\}$, where $(1_\Omega)_\omega \equiv \begin{cases} 1 & \text{for} \quad \omega \in \Omega \\ 0 & \text{for} \quad \omega \in \bar{\Omega}\backslash\Omega \end{cases}$

Unit Price Vector: $\mathbf{1}_\Omega^{-1} \equiv \left\{\left(1_\Omega^{-1}\right)_\omega; \omega \in \bar{\Omega}\right\}$, where $\left(1_\Omega^{-1}\right)_\omega \equiv \begin{cases} 1 & \text{for} \quad \omega \in \Omega \\ \infty & \text{for} \quad \omega \in \bar{\Omega}\backslash\Omega \end{cases}$

Note: $\int_\Omega (1_\Omega)_\omega d\omega = \int_\Omega \left(1_\Omega^{-1}\right)_\omega d\omega = |\Omega| \equiv V$.

At the symmetric patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$ and $\mathbf{x} = x\mathbf{1}_\Omega$,

$$s_\omega = s(1, \mathbf{p}/p_\omega) = s^*(1, \mathbf{x}/x_\omega) = s\left(1, \mathbf{1}_\Omega^{-1}\right) = s^*(1, \mathbf{1}_\Omega) = 1/V$$

$$\zeta_\omega = \zeta(1, \mathbf{p}/p_\omega) = \zeta^*(1, \mathbf{x}/x_\omega) = \zeta\left(1, \mathbf{1}_\Omega^{-1}\right) = \zeta^*(1, \mathbf{1}_\Omega) > 1$$

Clearly, this depends only on V. We propose:

Definition: *The substitutability measure across goods* is defined by

$$\mathcal{S}(V) \equiv \zeta\left(1; \mathbf{1}_\Omega^{-1}\right) = \zeta^*(1; \mathbf{1}_\Omega) > 1.$$

We call the case of $\mathcal{S}'(V) > (<) 0$ for all $V > 0$, the case of ***increasing (decreasing) substitutability***.

Notes:

- We can also define in terms of Allen-Uzawa Elasticity of Substitution evaluated at the symmetric patterns, which turns out to be equivalent.
- In general, the 2nd Law is neither sufficient nor necessary for increasing substitutability, $\mathcal{S}'(V) > 0$.

Love-for-Variety Measure: Commonly defined by the productivity gain from a higher V, holding xV

$$\left.\frac{d\ln X(\mathbf{x})}{d\ln V}\right|_{\mathbf{x}=x\mathbf{1}_\Omega, xV=const.} = \left.\frac{d\ln xX(\mathbf{1}_\Omega)}{d\ln V}\right|_{xV=const.} = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 > 0$$

Alternatively, LV may be defined by the decline in $P(\mathbf{p})$ from a higher V, at $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$, holding p constant.

$$-\left.\frac{d\ln P(\mathbf{p})}{d\ln V}\right|_{\mathbf{p}=p\mathbf{1}_\Omega^{-1},\ p=const.} = -\frac{d\ln P\left(\mathbf{1}_\Omega^{-1}\right)}{d\ln V} > 0.$$

Both are functions of V only, and equivalent because, by applying $\mathbf{x} = x\mathbf{1}_\Omega$ and $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$ to $\mathbf{px} = P(\mathbf{p})X(\mathbf{x})$,

$$pxV = pP\left(\mathbf{1}_\Omega^{-1}\right)xX(\mathbf{1}_\Omega) \Rightarrow -\frac{d\ln P\left(\mathbf{1}_\Omega^{-1}\right)}{d\ln V} = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 > 0.$$

Definition. *The love-for-variety measure* is defined by:

$$\mathcal{L}(V) \equiv -\frac{d\ln P\left(\mathbf{1}_\Omega^{-1}\right)}{d\ln V} = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 > 0.$$

We call the case of $\mathcal{L}'(V) < (>) 0$ for all $V > 0$, the case of ***diminishing (increasing) love-for-variety***.

Note: $\mathcal{L}(V) > 0$ is guaranteed by the strict quasi-concavity.

Standard CES with Gross Substitutes:

$$X(\mathbf{x}) = Z\left[\int_{\Omega} x_{\omega}{}^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \iff P(\mathbf{p}) = \frac{1}{Z}\left[\int_{\Omega} p_{\omega}{}^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}},$$

where $\sigma > 1$. ($Z > 0$ is TFP or affinity in the preference, in the context of spatial economics)

	CES
Budget Share	$s_{\omega} = \left(\frac{p_{\omega}}{ZP(\mathbf{p})}\right)^{1-\sigma} = \left(\frac{Zx_{\omega}}{X(\mathbf{x})}\right)^{1-1/\sigma}$
Price Elasticity	$\zeta_{\omega} = \sigma > 1$
Substitutability	$\mathcal{S}(V) = \sigma > 1$
Love-for-variety	$\mathcal{L}(V) = \frac{1}{\sigma - 1} > 0.$

Under Standard CES,

- PE of demand, $\zeta(p_{\omega}; \mathbf{p}) = \zeta^{*}(x_{\omega}; \mathbf{x})$, is independent of $\mathbf{p}$ or $\mathbf{x}$ and equal to σ.
- Substitutability, $\mathcal{S}(V)$, is independent of V and equal to σ.
- LV, $\mathcal{L}(V)$, is independent of $V,$ and equal to a constant, $\mathcal{L}(V) = \mathcal{L} = 1/(\sigma - 1)$, inversely related to σ.

General Homothetic Demand System: The relation btw $\zeta(p_{\omega}; \mathbf{p}) = \zeta^{*}(x_{\omega}; \mathbf{x})$, $\mathcal{S}(V)$, & $\mathcal{L}(V)$ can be complex.

- Whether the 2nd Law holds or not says little about the derivatives of $\mathcal{S}(V)$ and $\mathcal{L}(V)$.
- $\mathcal{S}(V)$ and $\mathcal{L}(V)$ could be positively related.

Digression: Generalized CES with Gross Substitutes a la Benassy (1996).

$$X(\mathbf{x}) = Z(V)\left[\int_{\Omega} x_{\omega}^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \Leftrightarrow P(\mathbf{p}) = \frac{1}{Z(V)}\left[\int_{\Omega} p_{\omega}^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}},$$

Note: $Z(V)$ allows variety to have direct externalities to TFP (or affinity)

	Under Generalized CES
Budget Share	$s_{\omega} = \left(\frac{p_{\omega}}{Z(V)P(\mathbf{p})}\right)^{1-\sigma} = \left(\frac{Z(V)x_{\omega}}{X(\mathbf{x})}\right)^{1-1/\sigma}$
Price Elasticity	$\zeta_{\omega} = \sigma > 1$
Substitutability	$\mathcal{S}(V) = \sigma > 1$
Love-for-variety	$\mathcal{L}(V) = \frac{1}{\sigma - 1} + \frac{d \ln Z(V)}{d \ln V}.$

- PE, ζ_{ω}, and Substitutability, $\mathcal{S}(V)$, are not affected by $d \ln Z(V)/d \ln V$, "the Benassy residual", which can "account for" the gap btw CES-implied LV (say, from the markup) & revealed LV (say, from productivity growth).
- Benassy (1996) set $d \ln Z(V)/d \ln V = \nu - 1/(\sigma - 1)$, so that $\mathcal{L}(V) = \nu$ is a separate parameter.

Even if you believe in the direct externalities behind the Benassy residual, your estimate of its magnitude depends on the CES assumption, which nobody believes.

In all the non-CES considered below, we could have let TFP vary directly with V, which would add the term, $d \ln Z(V)/d \ln V$, to the expression for $\mathcal{L}(V)$, without affecting the expression for $\mathcal{S}(V)$.

Geometric Means of CES

Two Versions of GM-CES

Let $G(\cdot)$ the cdf of $\sigma \in (1, \infty)$, and $\mathbb{E}_G[f(\sigma)]$: the expected value of $f(\sigma)$.

Weighted Geometric Means of Symmetric CES (GM-CES) Unit Cost Function

$$\ln P(\mathbf{p}) \equiv \int_1^\infty \ln P(\mathbf{p};\sigma)\, dG(\sigma) \equiv \mathbb{E}_G[\ln P(\mathbf{p};\sigma)] \qquad \text{where} \qquad [P(\mathbf{p};\sigma)]^{1-\sigma} \equiv \int_\Omega p_\omega{}^{1-\sigma}\, d\omega$$

Weighted Geometric Means of Symmetric CES (GM-CES) Production Function

$$\ln X(\mathbf{x}) \equiv \int_1^\infty \ln X(\mathbf{x};\sigma)\, dG(\sigma) \equiv \mathbb{E}_G[\ln X(\mathbf{x};\sigma)] \qquad \text{where} \qquad [X(\mathbf{x};\sigma)]^{1-\frac{1}{\sigma}} \equiv \int_\Omega x_\omega{}^{1-\frac{1}{\sigma}}\, d\omega$$

Clearly, both satisfy linear homogeneity, strict quasi-concavity, and symmetry.

	GM-CES Unit Cost Function	GM-CES Production Function
Budget Share	$s(p_\omega;\mathbf{p}) = \mathbb{E}_G\left[\left(\frac{p_\omega}{P(\mathbf{p};\sigma)}\right)^{1-\sigma}\right]$	$s^*(x_\omega;\mathbf{x}) = \mathbb{E}_G\left[\left(\frac{x_\omega}{X(\mathbf{x};\sigma)}\right)^{1-1/\sigma}\right]$
Price Elasticity	$\zeta(p_\omega;\mathbf{p}) = \frac{\mathbb{E}_G[\sigma p_\omega{}^{-\sigma}/[P(\mathbf{p};\sigma)]^{1-\sigma}]}{\mathbb{E}_G[p_\omega{}^{-\sigma}/[P(\mathbf{p};\sigma)]^{1-\sigma}]}$	$\zeta^*(x_\omega;\mathbf{x}) = \frac{\mathbb{E}_G[(x_\omega)^{-1/\sigma}/[X(\mathbf{x};\sigma)]^{1-1/\sigma}]}{\mathbb{E}_G[(x_\omega)^{-1/\sigma}/\sigma[X(\mathbf{x};\sigma)]^{1-1/\sigma}]}$
Substitutability	$\mathcal{S}(V) = \mathbb{E}_G[\sigma]$	$\mathcal{S}(V) = \frac{1}{\mathbb{E}_G[1/\sigma]}$
Love-for-Variety	$\mathcal{L}(V) = \mathbb{E}_G\left[\frac{1}{\sigma-1}\right] \geq \frac{1}{\mathcal{S}(V)-1}$	$\mathcal{L}(V) = \mathbb{E}_G\left[\frac{1}{\sigma-1}\right] \geq \frac{1}{\mathcal{S}(V)-1}$

Note: These GM-CES demand systems are *not* nested CES.

Theorem 1 (GM-CES): Under the two classes of GM-CES demand systems,
1-i): $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are both constant. For the GM-CES unit cost function,

$$\mathcal{S}(V) = \mathbb{E}_G[\sigma] > 1; \ \mathcal{L}(V) = \mathbb{E}_G\left[\frac{1}{\sigma - 1}\right] > 0.$$

For the GM-CES production function:

$$\mathcal{S}(V) = \frac{1}{\mathbb{E}_G[1/\sigma]} > 1; \ \mathcal{L}(V) = \mathbb{E}_G\left[\frac{1}{\sigma - 1}\right] > 0.$$

1-ii): $\mathcal{L}(V)$ can be arbitrarily large, without any upper bound, while its lower bound is given by:

$$\mathcal{L}(V) \geq \frac{1}{\mathcal{S}(V) - 1} > 0.$$

where the equality holds if and only if $G(\cdot)$ is degenerate, i.e., only under CES.

Notes:

- For a non-degenerate $G(\cdot)$, Jensen's inequality implies:

$$\mathcal{L}(V) - \frac{1}{\mathcal{S}(V) - 1} > 0; \qquad \mathbb{E}_G[\sigma] > \frac{1}{\mathbb{E}_G[1/\sigma]}$$

  - The 1st inequality may be interpreted as offering a microfoundation for the Benassy residual.
  - The CES formula for LV underestimates LV under GM-CES or thus overestimates the Benassy residual.
  - The 2^{nd} inequality implies that CES is the only intersection of the two classes of GM-CMS.

- There exist any number of families of cdf's, G, such that $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are positively related within each family.

H.S.A., HDIA, and HIIA

One might intuitively think that, as variety increases,

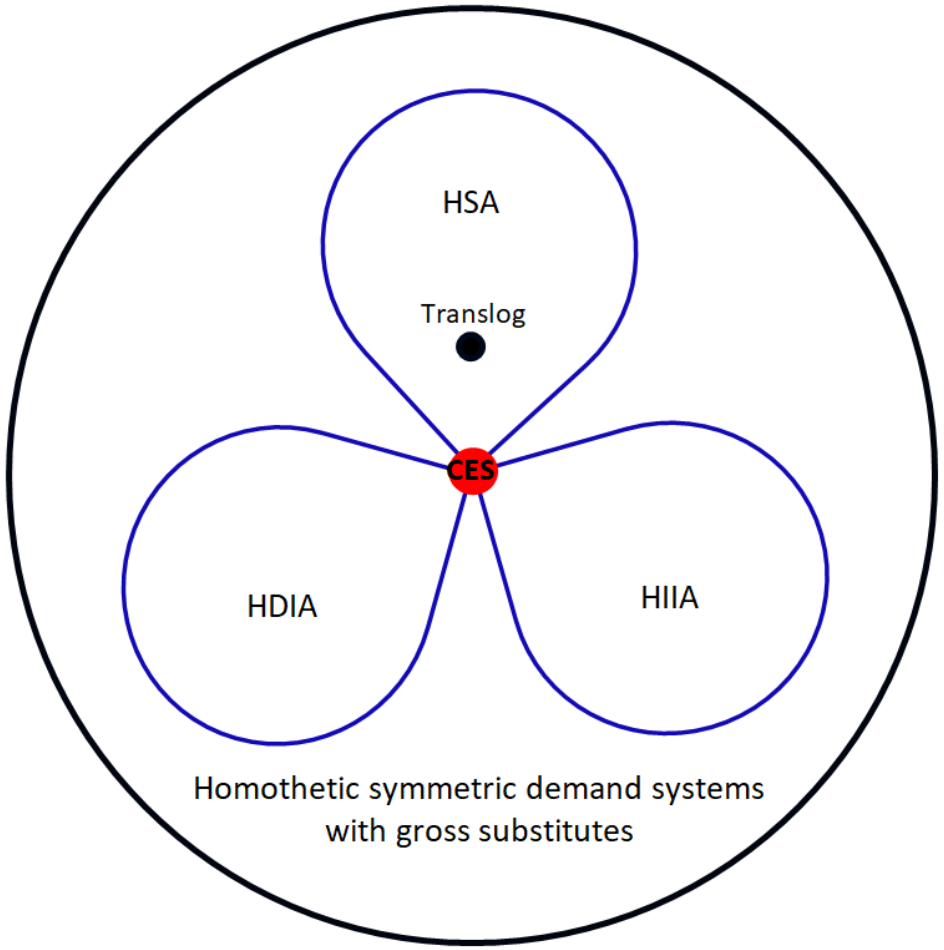
- PE of demand for each good become larger.
- Different goods become more substitutable
- LV becomes smaller.

Homotheticity is too general to capture this intuition!!
It is NOT restrictive enough.

To capture this intuition, we turn to

3 Classes of Symmetric Homothetic Demand System

✓ **Homothetic Single Aggregator (H.S.A.)**
✓ **Homothetic Direct Implicit Additivity (HDIA)**
✓ **Homothetic Indirect Implicit Additivity (HIIA)**



3 Classes of Symmetric Homothetic Demand Systems (with gross Substitutes & Inessentiality)

$\mathcal{M}[\cdot]$ is a monotone transformation.

Homothetic Direct Implicit Additivity (HDIA): $\phi(\cdot)$	$\mathcal{M}\left[\int_{\Omega} \phi\left(\frac{Zx_{\omega}}{X(\mathbf{x})}\right) d\omega\right] \equiv \mathcal{M}\left[\int_{\Omega} \phi\left(\frac{x_{\omega}}{\hat{X}(\mathbf{x})}\right) d\omega\right] \equiv 1.$

$\phi(\cdot)$: $\mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$, thus $\hat{X}(\mathbf{x})$, is independent of $Z > 0$, TFP.
$\phi(0) = 0; \phi(\infty) = \infty; \phi'(\infty) = 0; \phi'(y) > 0 > \phi''(y), 0 < -y\phi''(y)/\phi'(y) < 1$, for $\forall y \in (0, \infty)$.
CES with $\phi(y) = (y)^{1-1/\sigma}, \sigma > 1$. The choke price exists if $\phi'(0) < \infty$.

Homothetic Indirect Implicit Additivity (HIIA): $\theta(\cdot)$	$\mathcal{M}\left[\int_{\Omega} \theta\left(\frac{p_{\omega}}{ZP(\mathbf{p})}\right) d\omega\right] \equiv \mathcal{M}\left[\int_{\Omega} \theta\left(\frac{p_{\omega}}{\hat{P}(\mathbf{p})}\right) d\omega\right] \equiv 1.$

$\theta(\cdot)$: $\mathbb{R}_{++} \rightarrow \mathbb{R}_{+}$, thus $\hat{P}(\mathbf{p})$, is independent of $Z > 0$ is TFP.
$\theta(z) > 0, \theta'(z) < 0 < \theta''(z), -z\theta''(z)/\theta'(z) > 1$ for $0 < z < \bar{z} \leq \infty$, $\theta(0) = \infty; \theta(z) = \theta'(z) = 0$ for $z \geq \bar{z}$.
CES with $\theta(z) = (z)^{1-\sigma}, \sigma > 1$. The choke price exists if $\bar{z} < \infty$.

Homothetic Single Aggregator (H.S.A.): $s(\cdot)$	$s_{\omega} = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_{\omega}} = s\left(\frac{p_{\omega}}{A(\mathbf{p})}\right)$ with $\int_{\Omega} s\left(\frac{p_{\omega}}{A(\mathbf{p})}\right) d\omega \equiv 1.$

$s(\cdot)$: $\mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$, thus $A(\mathbf{p})$, is independent of $Z > 0$, TFP.
$s(z) > 0 > s'(z)$ for $0 < z < \bar{z} \leq \infty$; $s(z) = 0$ for $z \geq \bar{z}$. $s(0) = \infty$ to be well-defined for any arbitrarily small $V > 0$.
CES with $s(z) = \gamma z^{1-\sigma}, \sigma > 1$. The choke price exists if $\bar{z} < \infty$.
$Z > 0$ shows up when integrating the budget share to obtain $P(\mathbf{p})$ or $X(\mathbf{x})$.

Key Properties of the Three Classes

	Budget Shares: $s_\omega \equiv \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s(p_\omega; \mathbf{p})$		Price Elasticity: $\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega; \mathbf{p})$
CES	$s_\omega = \left(\frac{p_\omega}{ZP(\mathbf{p})}\right)^{1-\sigma}$		σ
H.S.A. $s(\cdot)$	$s_\omega = s\left(\frac{p_\omega}{A(\mathbf{p})}\right)$	$\frac{P(\mathbf{p})}{A(\mathbf{p})} \neq c$, unless CES	$\zeta^S\left(\frac{p_\omega}{A(\mathbf{p})}\right)$; $\zeta^S(z) \equiv 1 - \frac{zs'(z)}{s(z)} > 1$
HDIA $\phi(\cdot)$	$s_\omega = \frac{p_\omega}{P(\mathbf{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)$	$\frac{P(\mathbf{p})}{B(\mathbf{p})} \neq c$, unless CES	$\zeta^D\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right)$; $\zeta^D(y) \equiv -\frac{\phi'(y)}{y\phi''(y)} > 1$
HIIA $\theta(\cdot)$	$s_\omega = \frac{p_\omega}{C(\mathbf{p})}\theta'\left(\frac{p_\omega}{P(\mathbf{p})}\right)$	$\frac{P(\mathbf{p})}{C(\mathbf{p})} \neq c$, unless CES	$\zeta^I\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)$; $\zeta^I(z) \equiv -\frac{z\theta''(z)}{\theta'(z)} > 1.$

$A(\mathbf{p}), B(\mathbf{p}), C(\mathbf{p})$: each defined implicitly by the adding-up constraint, $\int_\Omega s_\omega d\omega \equiv 1$. Clearly, they are all linear homogenous.

We focus on these three classes for two reasons.

- They are pairwise disjoint with the sole exception of CES.
- PE $= \zeta_\omega \equiv \zeta\left(\frac{p_\omega}{\mathcal{A}(\mathbf{p})}\right)$, where $\mathcal{A}(\mathbf{p})$ is linear homogenous, a sufficient statistic, capturing all the cross-product effects.

Key Properties of the Three Classes, Continued.

	Price Elasticity: $\zeta(p_\omega;\mathbf{p})$	**Substitutability :** $\mathcal{S}(V)$	**Love-for-Variety:** $\mathcal{L}(V)$
H.S.A.	$\zeta_\omega = \zeta^S\left(\frac{p_\omega}{A(\mathbf{p})}\right)$	$\zeta^S\left(s^{-1}\left(\frac{1}{V}\right)\right)$	$\Phi\left(s^{-1}\left(\frac{1}{V}\right)\right) = \frac{1}{\mathcal{E}_H\left(s^{-1}(1/V)\right)}$,

where $\zeta^S(z) \equiv -\frac{zH''(z)}{H'(z)} > 1$ and $\frac{1}{\Phi(z)} = \mathcal{E}_H(z) \equiv -\frac{zH'(z)}{H(z)} > 0$, with $H(z) \equiv \int_z^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi > 0$.

HDIA	$\zeta_\omega = \zeta^D\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right)$	$\zeta^D\left(\phi^{-1}\left(\frac{1}{V}\right)\right)$	$\frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/V))} - 1$

where $\zeta^D(y) \equiv -\frac{\phi'(y)}{y\phi''(y)} > 1$ and $0 < \mathcal{E}_\phi(y) \equiv \frac{y\phi'(y)}{\phi(y)} < 1$.

HIIA	$\zeta_\omega = \zeta^I\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)$	$\zeta^I\left(\theta^{-1}\left(\frac{1}{V}\right)\right)$	$\frac{1}{\mathcal{E}_\theta\left(\theta^{-1}(1/V)\right)}$

where $\zeta^I(z) \equiv -\frac{z\theta''(z)}{\theta'(z)} > 1$ and $\mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} > 0$.

Note: In all three classes,

- $\mathcal{L}(V)$ depends on the curvature of a function of a single variable, $H(\cdot), \phi(\cdot), \theta(\cdot)$
- $\mathcal{S}(V)$ depends on the curvature of its derivative. $H'(\cdot), \phi'(\cdot), \theta'(\cdot)$.

Theorem 2: Under H.S.A., HDIA, & HIIA,
2-i) $\mathcal{S}'(V) > 0$ iff the 2nd law holds.
2-ii) $\mathcal{S}'(V) \gtreqless 0$ for all $V \in (V_0, \infty) \Longrightarrow \mathcal{L}'(V) \lesseqgtr 0$ for all $V \in (V_0, \infty)$.
The converse is not true in general. However,
2-iii) $\mathcal{L}'(V) = 0$ for all $V \in (V_0, \infty) \Leftrightarrow \mathcal{S}'(V) = 0$ for all $V \in (V_0, \infty)$.
In particular, $\mathcal{L}'(V) = 0$ for all $V > 0 \Leftrightarrow \mathcal{S}'(V) = 0$ for all $V > 0 \Leftrightarrow$ CES.

Theorem 3: Under H.S.A., HDIA, & HIIA,

$$\mathcal{L}'(V) \lesseqgtr 0 \Leftrightarrow \mathcal{L}(V) \lesseqgtr \frac{1}{\mathcal{S}(V) - 1} > 0$$

The 2nd Law
$\zeta(p_\omega; \mathbf{p})$ is increasing in p_ω
$\zeta^*(x_\omega; \mathbf{x})$ is decreasing in x_ω

$\Updownarrow$

Increasing Substitutability
$\mathcal{S}'(V) > 0$ for all $V > 0$.

$\Rightarrow$

Diminishing Love-for-Variety
$\mathcal{L}'(V) < 0$ for all $V > 0$.

$\Leftrightarrow$

The CES formula overestimates Love-for-Variety.
$$\mathcal{L}(V) < \frac{1}{\mathcal{S}(V) - 1}$$

Theorem 4: Under H.S.A., HDIA, & HIIA, $\lim_{V\to\infty} \mathcal{L}(V) = \lim_{V\to\infty} \frac{1}{\mathcal{S}(V)-1}$. In particular, $\lim_{V\to\infty} \mathcal{S}(V) = \infty \Leftrightarrow \lim_{V\to\infty} \mathcal{L}(V) = 0$.

An Application to an Armington Model of Trade

Armington Model of Competitive Trade:

Two Countries: Home & Foreign* differ only in labor supply L & L^*(with the wage rates, w & w^*) and goods they produce, Ω & Ω^*; $\Omega \cap \Omega^* = \emptyset$, with $V \equiv |\Omega|$ & $V^* \equiv |\Omega^*|$.

Technology: One unit of Home (Foreign) labor produces one unit of each Home (Foreign) good.

No Trade Cost: In both countries, the unit prices of goods are $p_\omega = w$ $(\omega \in \Omega)$ and $p_\omega^* = w^*$ $(\omega \in \Omega^*)$.

Symmetric Homothetic Demand:
D &M: Home demand for each Home & Foreign good; D^* & M^*: Foreign demand for each Foreign & Home good.

	Home	**Foreign**
Resource Constraint:	$V(D + M^*) = L$	$V^*(M + D^*) = L^*$
Budget Constraint:	$wVD + w^*V^*M = wL$	$wVM^* + w^*V^*D^* = w^*L^*.$
Trade-GDP Ratio:	$\frac{w^*V^*M}{wL} = \frac{wVM^*}{wL} = \frac{VM^*}{L}$	$\frac{wVM^*}{w^*L^*} = \frac{w^*V^*M}{w^*L^*} = \frac{V^*M}{L^*}$

Balanced Trade	$wVM^* = w^*V^*M.$
Relative Supply = Relative Demand:	$\frac{L/V}{L^*/V^*} = RS = RD = \frac{D + M^*}{M + D^*} = \frac{D}{M} = \frac{M^*}{D^*} = g\left(\frac{w}{w^*}; V; V^*\right) \gtreqless 1 \Leftrightarrow \frac{w}{w^*} \lesseqgtr 1.$

Consider the case where the two countries differ proportionally in size with f being the Home's share.

$$\frac{L}{L^*} = \frac{V}{V^*}$$

Then, from the RS = RD condition,

$$\frac{L/V}{L^*/V^*} = 1 \Leftrightarrow \frac{w}{w^*} = 1 \Leftrightarrow \frac{D}{M} = \frac{M^*}{D^*} = 1.$$

Balanced Trade condition becomes $VM^* = V^*M$.

$$\frac{L}{L^*} = \frac{V}{V^*} = \frac{M}{M^*} = \frac{D}{D^*} \Leftrightarrow \frac{V}{L} = \frac{V^*}{L^*};\ \frac{D}{L} = \frac{D^*}{L^*} = \frac{M}{L} = \frac{M^*}{L^*}.$$

Per capita term, Home and Foreign become identical.

	Home	**Foreign**
Domestic Expenditure Share	$\lambda = \dfrac{V}{V + V^*}$	$\lambda^* = \dfrac{V^*}{V + V^*}$
Trade-GDP Ratio	$1 - \lambda = \dfrac{V^*}{V + V^*}$	$1 - \lambda^* = \dfrac{V}{V + V^*}$

Gains from Trade: Equivalent to $V \rightarrow\ V + V^* = V/\lambda$ for Home and to $V^* \rightarrow V + V^* = V^*/\lambda^*$ for Foreign.

	Home	**Foreign**
Gains from Trade	$GT \equiv \dfrac{P\left(\mathbf{1}_{\Omega}^{-1}\right)}{P(\mathbf{1}_{\Omega\cup\Omega^*}^{-1})} = \exp\left[\int_{V}^{V/\lambda} \mathcal{L}(v)\dfrac{dv}{v}\right]$	$GT^* \equiv \dfrac{P\left(\mathbf{1}_{\Omega^*}^{-1}\right)}{P(\mathbf{1}_{\Omega\cup\Omega^*}^{-1})} = \exp\left[\int_{V^*}^{V^*/\lambda^*} \mathcal{L}(v)\dfrac{dv}{v}\right]$

The effect of Home openness, $\lambda = V/(V + V^*) \downarrow$, on Home GT may depend on whether it is due to $V \downarrow$ or $V^* \uparrow$.

General Implications:

Theorem 5 (The Effects of Country Sizes, V and V^* on Gains from Trade):

5-i) GT is larger for the smaller country than for the larger country.

$$GT \gtreqless GT^* \Leftrightarrow V \lesseqgtr V^* \Leftrightarrow \lambda \lesseqgtr \lambda^*$$

5-ii) If the two countries are proportionately larger, GT are diminishing for both countries under diminishing LV.

$$\left.\frac{\partial \ln(GT)}{\partial \ln V}\right|_{\lambda=const.} = \mathcal{L}(V/\lambda) - \mathcal{L}(V) < 0; \qquad \left.\frac{\partial \ln(GT^*)}{\partial \ln V^*}\right|_{\lambda^*=const.} = \mathcal{L}(V^*/\lambda^*) - \mathcal{L}(V^*) < 0;$$

5-iii) For any given V, GT is increasing in V^* (thus decreasing in λ),

$$\left.\frac{\partial \ln(GT)}{\partial \ln V^*}\right|_{V=const.} = (1-\lambda)\mathcal{L}(V/\lambda) > 0$$

with the range,

$$0 < \ln(GT) < \int_V^\infty \mathcal{L}(v)\frac{dv}{v}.$$

The upper bound is infinite if $\mathcal{L}(\infty) > 0$. It may be finite if $\mathcal{L}(\infty) = 0$. If finite, the upper bound is decreasing in V.

5-iv) For any given V^*, GT may be nonmonotone in V (thus in λ) in general. Under non-increasing LV,

$$\left.\frac{\partial \ln(GT)}{\partial \ln V}\right|_{V^*=const.} = \lambda\mathcal{L}(V/\lambda) - \mathcal{L}(V) < 0$$

hence GT is decreasing in V (thus in λ), with the range

$$0 < \ln(GT) < \int_0^{V^*} \mathcal{L}(v)\frac{dv}{v}.$$

The upper bound is finite if $\mathcal{L}(0) < \infty$. It may be infinite if $\mathcal{L}(0) = \infty$. If finite, the upper bound is increasing in V^*.

Gains from Trade under CES

Substitutability: $\mathcal{S}(V)$	Love-for-Variety: $\mathcal{L}(V)$	Home Gains from Trade
$\mathcal{S}^{CES} = \sigma > 1$	$\mathcal{L}^{CES} = \frac{1}{\sigma - 1}$	$\ln(GT) = \mathcal{L}^{CES} \ln\left(\frac{1}{\lambda}\right) = \frac{1}{\mathcal{S}^{CES} - 1} \ln\left(\frac{1}{\lambda}\right)$

- $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are constant under CES.
- GT satisfies the familiar ACR formula.
- Decreasing in λ (thus increasing in the openness, $1 - \lambda$).
- GT goes to infinity as $\lambda \to 0$, or $V/V^* \to 0$. Once λ is controlled for, V and V^* play no role.

Gains from Trade under GM-CES

Substitutability: $\mathcal{S}(V)$	Love-for-Variety: $\mathcal{L}(V)$	Home Gains from Trade
$\mathcal{S}^{GMCES} = \mathbb{E}_G[\sigma]$ for GM-CES unit cost fn. $\mathcal{S}^{GMCES} = \left[\mathbb{E}_G[1/\sigma]\right]^{-1}$ for GM-CES production fn.	$\mathcal{L}^{GMCES} = \mathbb{E}_G\left[\frac{1}{\sigma - 1}\right]$	$\ln(GT) = \mathcal{L}^{GMCES} \ln\left(\frac{1}{\lambda}\right) \geq \frac{1}{\mathcal{S}^{GMCES} - 1} \ln\left(\frac{1}{\lambda}\right)$

- $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are also constant under GM-CES.
- GT satisfies the familiar ACR formula, with $\mathcal{L}^{GMCES}$ but not with $\mathcal{S}^{GMCES}$.
- GT is decreasing in λ (thus increasing in the openness, $1 - \lambda$).
- GT goes to infinity as $\lambda \to 0$, or $V/V^* \to 0$. Once λ is controlled for, V and V^* play no role.
- For any level of λ, GT under GM-CES can be arbitrarily large, with GT under CES being the lower bound.

If one views $\mathcal{S}^{GMCES}$ being constant as the evidence for CES, one would underestimate GT under GM-CES.

Gains from Trade under H.S.A.

Substitutability: $\mathcal{S}(V)$	Love-for-Variety: $\mathcal{L}(V)$	Home Gains from Trade
$\zeta^S\left(s^{-1}\left(\frac{1}{V}\right)\right)$	$\Phi\left(s^{-1}\left(\frac{1}{V}\right)\right)$	$GT = \frac{s^{-1}(\lambda/V)}{s^{-1}(1/V)} \frac{\exp\left[\Phi\left(s^{-1}(\lambda/V)\right)\right]}{\exp\left[\Phi\left(s^{-1}(1/V)\right)\right]}$

- For a given V, $V^* \uparrow$ (thus $\lambda \downarrow$) increases Home GT, with the upper bound

$$GT < \frac{\bar{z}}{s^{-1}(1/V)} \frac{\exp[\Phi(\bar{z})]}{\exp\left[\Phi\left(s^{-1}(1/V)\right)\right]} < \infty \iff \bar{z} < \infty.$$

If finite, the upper bound is decreasing in V. CES & GM-CES overestimate gains from trade with a large country.

- For a given V^*, $V \downarrow$ (thus $\lambda \downarrow$) increases Home GT, under non-increasing LV. The upper bound is infinite.

$$GT < \frac{s^{-1}(1/V^*)}{s^{-1}(\infty)} \frac{\exp[\Phi(1/V^*)]}{\exp\left[\Phi\left(s^{-1}(\infty)\right)\right]} = \infty.$$

Parametric Examples of H.S.A. All feature the 2nd law, Increasing Substitutability, Diminishing LV, the choke price.

	$\mathcal{S}(V)$	Love-for-Variety: $\mathcal{L}(V)$	Home Gains from Trade
Translog	$1 + \gamma V$	$\frac{1/2}{\gamma V}$	$\ln(GT) = \frac{1-\lambda}{2\gamma V}$
Generalized Translog	$1 + (\sigma - 1)(\gamma V)^{1/\eta}$	$\frac{\eta/(1+\eta)}{(\sigma-1)(\gamma V)^{1/\eta}}$	$\ln(GT) = \frac{(\gamma V)^{-1/\eta}}{\sigma - 1} \frac{\eta}{1+\eta} \frac{1-(\lambda)^{1/\eta}}{1/\eta}$
CoPaTh	$\sigma(\gamma V)^{\frac{1-\rho}{\rho}} > 1$	$\sum_{n=0}^{\infty} \frac{\rho}{1+(1-\rho)n}\left[\frac{1}{\sigma(\gamma V)^{\frac{1-\rho}{\rho}}}\right]^n$	$\ln(GT) = -\sum_{n=0}^{\infty} \frac{\rho}{1+(1-\rho)n} \frac{1-(\lambda)^{\frac{1-\rho}{\rho}(n+1)}}{\left[\sigma(\gamma V)^{\frac{1-\rho}{\rho}}\right]^{(n+1)}} + \ln\left[1 + \frac{1-(\lambda)^{\frac{1-\rho}{\rho}}}{\sigma(\gamma V)^{\frac{1-\rho}{\rho}} - 1}\right]^{\frac{\rho}{1-\rho}}$

Generalized Translog $(0 < \eta < \infty)$**:** The case of $\eta = 1$ is isomorphic to Translog. CES is the limit case, $\eta \to \infty$.
CoPaTh $(0 < \rho < 1)$**:** CES is the limit case, $\rho \to 1$.

	Substitutability: $\mathcal{S}(V)$	Love-for-Variety: $\mathcal{L}(V)$	Gains from Trade
HDIA	$\zeta^D\left(\phi^{-1}\left(\frac{1}{V}\right)\right)$	$\frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/V))} - 1$	$GT = \frac{(\lambda/V)}{\phi^{-1}(\lambda/V)}\frac{\phi^{-1}(1/V)}{(1/V)}$

- For a given V, $V^* \uparrow$ (thus $\lambda \downarrow$) increases Home GT, with the upper bound

$$GT < \phi'(0)\frac{\phi^{-1}(1/V)}{(1/V)} < \infty \Leftrightarrow \phi'(0) < \infty.$$

If finite. the upper bound is decreasing in V. CES and GM-CES overestimate gains from trade with a large country.

- For a given V^*, $V \downarrow$ (thus $\lambda \downarrow$) increases Home GT, under non-increasing LV. The upper bound is infinite.

$$GT < \frac{(1/V^*)}{\phi^{-1}(1/V^*)}\frac{1}{\phi'(\infty)} = \infty.$$

	Substitutability: $\mathcal{S}(V)$	Love-for-Variety: $\mathcal{L}(V)$	Gains from Trade
HIIA	$\mathcal{S}^I(V) = \zeta^I\left(\theta^{-1}\left(\frac{1}{V}\right)\right)$	$\mathcal{L}^I(V) = \frac{1}{\mathcal{E}_\theta\left(\theta^{-1}(1/V)\right)}$	$GT = \frac{\theta^{-1}(\lambda/V)}{\theta^{-1}(1/V)}$

- For a given V, $V^* \uparrow$ (thus $\lambda \downarrow$) increases Home GT, with the upper bound

$$GT < \frac{\bar{z}}{\theta^{-1}(1/V)} < \infty \Leftrightarrow \bar{z} < \infty.$$

If finite. the upper bound is decreasing in V. CES and GM-CES overestimate gains from trade with a large country.

- For a given V^*, $V \downarrow$ (thus $\lambda \downarrow$) increases Home GT, under non-increasing LV. The upper bound is infinite.

$$GT < \frac{\theta^{-1}(1/V^*)}{\theta^{-1}(\infty)} = \infty.$$

Concluding Remarks

What We Did in This Paper

- We investigated how LV depends on the underlying demand structure outside of CES.
- We defined **Substitutability & Love-for-Variety (LV),** a function of V only under homotheticity & symmetry
- **GM-CES:** The CES formula would underestimate LV under GM-CES (and overestimate the Benassy residuals).
- **3 classes (H.S.A., HDIA, HIIA):**
  - 2nd Law ⟺ Increasing Substitutability ⟹ Diminishing LV⟹ The CES formula would overestimate LV (and underestimate Benassy residuals)
  - LV goes asymptotically to zero, as V goes to infinity, if the choke price exists
- We illustrated some implications on gains from trade (GT) in a simple Armington model of trade.
  - GM-CES: Though ACR formula holds, CES underestimate GT, controlling for the openness.
  - H.S.A. HDIA and HIIA with the choke price. GT is increasing in the size of the trading partner, but it is bounded, unlike CES. CES may overestimate gains from trade with a large country.

Other Applications

- **Implications on Gravity Law:** using Armington models of trade with iceberg trade costs.
- **Static Monopolistic Competition:** Under GM-CES, insufficient entry. Under all 3 classes, the 2^{nd} Law ⟺ Procompetitive Entry ⟹ Excessive Entry, as shown in Matsuyama-Ushchev (2020), which we need to revise.
- **Romer-type Endogenous Growth with Expanding Variety/Knowledge Spillover**
  - Under CES and GM-CES, too little R&D in equilibrium.
  - Under the 3 classes with the 2^{nd} law, R&D can be too much in equilibrium, as in a vertical innovation model.

Appendices

Homothetic Single Aggregator (H.S.A.)

Symmetric H.S.A. (Homothetic Single Aggregator) DS with Gross Substitutes

Definition: A symmetric CRS technology, $P = P(\mathbf{p})$ is called *homothetic single aggregator* (H.S.A.) if the budget share of ω depends solely on a single variable, $z_\omega \equiv p_\omega / A$, its own price p_ω, normalized by the common price aggregator, $A = A(\mathbf{p})$.

$$s_\omega \equiv \frac{p_\omega x_\omega}{\mathbf{px}} = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s\left(\frac{p_\omega}{A(\mathbf{p})}\right), \qquad \text{where} \qquad \int_\Omega s\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega \equiv 1.$$

- $s: \mathbb{R}_{++} \to \mathbb{R}_+$: **the budget share function**, C^2, decreasing in the normalized price, $z_\omega \equiv p_\omega / A$ for $s(z_\omega) > 0$ with
  - $\lim_{z \to \bar{z}} s(z) = 0$. If $\bar{z} \equiv \inf\{z > 0 | s(z) = 0\} < \infty$, $\bar{z}A(\mathbf{p})$ is **the choke price.**
- $A = A(\mathbf{p})$: **the common price aggregator**, defined implicitly by **the adding-up constraint**, $\int_\Omega s(p_\omega / A) d\omega \equiv 1$.

By construction, the budget shares add up to one. $A(\mathbf{p})$ linear homogenous in $\mathbf{p}$ for a fixed Ω. A larger Ω reduces $A(\mathbf{p})$.
Some Special Cases

CES with gross substitutes	$s(z) = \gamma(z)^{1-\sigma}$	$\sigma > 1$
Generalized Translog (GTL)	$s(z) = \gamma\left(1 - \frac{\sigma-1}{\eta}\ln(z)\right)^\eta = \gamma\left(-\frac{\sigma-1}{\eta}\ln\left(\frac{z}{\bar{z}}\right)\right)^\eta$	$z < \bar{z} \equiv e^{\frac{\eta}{\sigma-1}} < \infty$
Standard Translog for $\eta = 1$. CES is the limit case, as $\eta \to \infty$, while holding $\sigma > 1$ fixed, so that $\bar{z} \equiv e^{\frac{\eta}{\sigma-1}} \to \infty$.		
Constant Pass Through (CoPaTh)	$s(z) = \gamma\left[\sigma - (\sigma-1)(z)^{\frac{1-\rho}{\rho}}\right]^{\frac{\rho}{1-\rho}} = \gamma\sigma^{\frac{\rho}{1-\rho}}\left[1 - \left(\frac{z}{\bar{z}}\right)^{\frac{1-\rho}{\rho}}\right]^{\frac{\rho}{1-\rho}}$	$\sigma > 1;\ 0 < \rho < 1$
	As $\rho \nearrow 1$, CoPaTh converges to CES with $\bar{z} = \left(\frac{\sigma}{\sigma-1}\right)^{\frac{\rho}{1-\rho}} \to \infty$.	

Price Elasticity:	$\zeta_\omega = \zeta(p_\omega; \mathbf{p}) = 1 - \frac{z_\omega s'(z_\omega)}{s(z_\omega)} \equiv \zeta^S(z_\omega) > 1$

Notes:

- A function of a single variable, $z_\omega \equiv p_\omega / A(\mathbf{p})$.
- $\zeta^S(z_\omega) = \sigma > 1$ under CES, $s(z) = \gamma z^{1-\sigma}$.
- The 2nd law iff $\zeta^{S\prime}(\cdot) > 0$, e.g., $\zeta^S(z_\omega) = 1 - \frac{\eta}{\ln(z_\omega/\bar{z})}$ for GTL; $\zeta^S(z_\omega) = \frac{1}{1-(z_\omega/\bar{z})^{(1-\rho)/\rho}}$ for CoPaTh.

Unit Cost Function: By integrating $\frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s\left(\frac{p_\omega}{A(\mathbf{p})}\right)$,

$$\ln\left[\frac{A(\mathbf{p})}{cP(\mathbf{p})}\right] = \int_\Omega \left[\int_{p_\omega/A(\mathbf{p})}^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi\right] d\omega \equiv \int_\Omega H\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega \equiv \int_\Omega s\left(\frac{p_\omega}{A(\mathbf{p})}\right) \Phi\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega,$$

where $c > 0$ is a constant, proportional to TFP. and $H(z)$ is decreasing and convex

$$H(z) \equiv \int_z^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi > 0 \implies H'(z) = -\frac{s(z)}{z} < 0 \implies \frac{zH''(z)}{H'(z)} = \frac{zs'(z)}{s(z)} - 1 < -1.$$

$$\Phi(z) \equiv \frac{1}{s(z)} \int_z^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi = \frac{H(z)}{s(z)} = -\frac{H(z)}{zH'(z)} > 0$$

$$\zeta^S(z) \equiv 1 - \frac{zs'(z)}{s(z)} \equiv -\frac{zH''(z)}{H'(z)} > 1.$$

Notes:

- $P(\mathbf{p})$: linear homogeneous, monotonic, and strictly quasi-concave, ensuring the integrability of H.S.A.
- $A(\mathbf{p})/P(\mathbf{p})$ is not constant and depends on $\mathbf{p}$, with the sole exception of CES, because

$$\frac{\partial \ln A(\mathbf{p})}{\partial \ln p_\omega} = \frac{z_\omega s'(z_\omega)}{\int_\Omega s'(z_{\omega'}) z_{\omega'} d\omega'} = \frac{[\zeta^S(z_\omega) - 1]s(z_\omega)}{\int_\Omega [\zeta^S(z_{\omega'}) - 1]s(z_{\omega'}) d\omega'} \neq \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s(z_\omega),$$

unless $\zeta^S(z)$ is constant, i.e., $\zeta^S(z) = \sigma > 1 \Leftrightarrow s(z) = \gamma z^{1-\sigma} \Leftrightarrow \Phi(z) = 1/(\sigma - 1) \Leftrightarrow H(z) = \gamma z^{1-\sigma}/(\sigma - 1)$.

For symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$,

$$1 = s\left(\frac{p_\omega}{A(\mathbf{p})}\right) V = s\left(\frac{1}{A(\mathbf{1}_\Omega^{-1})}\right) V \Rightarrow \frac{1}{A(\mathbf{1}_\Omega^{-1})} = s^{-1}\left(\frac{1}{V}\right).$$

$$-\ln cP(\mathbf{1}_\Omega^{-1}) = \ln s^{-1}\left(\frac{1}{V}\right) + \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right)$$

$$\Rightarrow -\frac{d \ln P(\mathbf{1}_\Omega^{-1})}{d \ln V} = \left.\frac{d[\ln z + \Phi(z)]}{d \ln(1/s(z))}\right|_{z=s^{-1}(1/V)} = -\left.\left[\frac{d[\ln z + \Phi(z)]}{d \ln z} \Big/ \frac{d \ln s(z)}{d \ln z}\right]\right|_{z=s^{-1}(1/V)} = \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right).$$

	H.S.A.
Substitutability	$\mathcal{S}^S(V) \equiv \zeta^S\left(s^{-1}\left(\frac{1}{V}\right)\right)$, where $\zeta^S(z) \equiv 1 - \frac{zs'(z)}{s(z)} \equiv -\frac{zH''(z)}{H'(z)} > 1.$
Love-for-variety	$\mathcal{L}^S(V) \equiv = \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right)$, where $\Phi(z) \equiv \frac{1}{s(z)}\int_z^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi = -\frac{H(z)}{zH'(z)} > 0.$

Notes:

Since $s^{-1}(1/V)$ is increasing in V,

$$\zeta^{S\prime}(\cdot) \gtreqless 0 \Leftrightarrow \mathcal{S}^{S\prime}(\cdot) \gtreqless 0.$$
$$\Phi'(\cdot) \gtreqless 0 \Leftrightarrow \mathcal{L}^{S\prime}(\cdot) \gtreqless 0.$$

Furthermore,

$$\frac{1}{\Phi(z)} = -\frac{zH'(z)}{H(z)} = \frac{\bar{z}H'(\bar{z}) - zH'(z)}{H(z)} = \frac{\int_z^{\bar{z}} d[\xi H'(\xi)]}{H(z)} = \frac{\int_z^{\bar{z}} [\xi H''(\xi) + H'(\xi)]\, d\xi}{H(z)} = \int_z^{\bar{z}} [\zeta^S(\xi) - 1] w(\xi; z)\, d\xi,$$

where $w(\xi; z) \equiv -H'(\xi)/H(z) > 0$, which satisfies $\int_z^{\bar{z}} w(\xi; z)\, d\xi = 1$. Hence,

$$\frac{z\Phi'(z)}{\Phi(z)} = \frac{zH'(z)}{H(z)} - 1 - \frac{zH''(z)}{H'(z)} = \zeta^S(z) - 1 - \frac{1}{\Phi(z)} = \zeta^S(z) - \int_z^{\bar{z}} \zeta^S(\xi) w(\xi; z)\, d\xi.$$

Hence,

Proposition S-1: $$\frac{z\Phi'(z)}{\Phi(z)} = \zeta^S(z) - 1 - \frac{1}{\Phi(z)} = \zeta^S(z) - \int_z^{\bar{z}} \zeta^S(\xi) w(\xi; z) d\xi.$$ where $w^S(\xi; z) \equiv -H'(\xi)/H(z)$, which satisfies $\int_z^{\bar{z}} w^S(\xi; z) d\xi = 1$. Hence, $$\zeta^{S\prime}(z) \gtreqless 0, \forall z \in (z_0, \bar{z}) \implies \Phi'(z) \lesseqgtr 0, \forall z \in (z_0, \bar{z}).$$ The opposite is not true in general. However, $$\zeta^{S\prime}(z) = 0, \forall z \in (z_0, \bar{z}) \Leftrightarrow \Phi'(z) = 0, \forall z \in (z_0, \bar{z}).$$
Proposition S-2. For $s(z_0)V_0 = 1$, $$\zeta^{S\prime}(z) \gtreqless 0, \forall z \in (z_0, \bar{z}) \Leftrightarrow \mathcal{S}^{S\prime}(V) \gtreqless 0, \forall V \in (V_0, \infty);$$ $$\Phi'(z) \lesseqgtr 0, \forall z \in (z_0, \bar{z}) \Leftrightarrow \mathcal{L}^{S\prime}(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$ Moreover, $$\mathcal{S}^{S\prime}(V) \gtreqless 0, \forall V \in (V_0, \infty) \Rightarrow \mathcal{L}^{S\prime}(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$ The opposite is not true in general. However, $$\mathcal{S}^{S\prime}(V) = 0, \forall V \in (V_0, \infty) \Leftrightarrow \mathcal{L}^{S\prime}(V) = 0, \forall V \in (V_0, \infty).$$
Proposition S-3. $$\mathcal{L}'(V) \lesseqgtr 0 \Leftrightarrow \frac{z\Phi'(z)}{\Phi(z)} = \zeta^S(z) - 1 - \frac{1}{\Phi(z)} \lesseqgtr 0 \Leftrightarrow \mathcal{L}(V) \lesseqgtr \frac{1}{\mathcal{S}(V) - 1}.$$
Proposition S-4. $$\lim_{V\to\infty} \mathcal{L}(V) = \lim_{z\to\bar{z}} \left(-\frac{s(z)}{zs'(z)}\right) = \lim_{z\to\bar{z}} \frac{1}{\zeta^S(z) - 1} = \lim_{V\to\infty} \frac{1}{\mathcal{S}(V) - 1}.$$

Gains from Trade under H.S.A.

$$\ln(GT) = \int_V^{V+V^*} \mathcal{L}^S(v)\frac{dv}{v} = \int_V^{V/\lambda} \Phi\left(s^{-1}\left(\frac{1}{v}\right)\right)\frac{dv}{v}.$$

Using the change of variables, $s^{-1}\left(\frac{1}{v}\right) = \xi \Leftrightarrow v = \frac{1}{s(\xi)}$, which implies $\frac{dv}{v} = -\frac{\xi s'(\xi)}{s(\xi)}\frac{d\xi}{\xi} = (\zeta^S(\xi) - 1)\frac{d\xi}{\xi}$,

and the identity, $\Phi(\xi)(\zeta^S(\xi) - 1) = 1 + \xi\Phi'(\xi)$,

$$\ln(GT) = \int_V^{V/\lambda} \Phi\left(s^{-1}\left(\frac{1}{v}\right)\right)\frac{dv}{v} = \int_{s^{-1}(1/V)}^{s^{-1}(\lambda/V)} \Phi(\xi)(\zeta^S(\xi) - 1)\frac{d\xi}{\xi} = \int_{s^{-1}(1/V)}^{s^{-1}(\lambda/V)} [1 + \xi\Phi'(\xi)]\frac{d\xi}{\xi}$$

$$= \int_{s^{-1}(1/V)}^{s^{-1}(\lambda/V)} \frac{d\xi}{\xi} + \int_{s^{-1}(1/V)}^{s^{-1}(\lambda/V)} \Phi'(\xi)d\xi = \ln\left[\frac{s^{-1}(\lambda/V)}{s^{-1}(1/V)}\right] + \Phi\left(s^{-1}\left(\frac{\lambda}{V}\right)\right) - \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right).$$

from which,

$$GT = \frac{s^{-1}(1/(V + V^*))}{s^{-1}(1/V)}\frac{\exp[\mathcal{L}^S((V + V^*)]}{\exp[\mathcal{L}^S(V)]} = \frac{s^{-1}(\lambda/V)}{s^{-1}(1/V)}\frac{\exp[\mathcal{L}^S(V/\lambda)]}{\exp[\mathcal{L}^S(V)]}.$$

Parametric Examples of H.S.A.

Generalized Translog (GTL): $s(z) = \gamma\left(\frac{\sigma-1}{\eta}\ln\left(\frac{\overline{z}}{z}\right)\right)^{\eta}$; $\overline{z} \equiv \beta e^{\frac{\eta}{\sigma-1}} \Rightarrow z = s^{-1}\left(\frac{1}{V}\right) = \overline{z}\exp\left\{-\frac{\eta}{(\sigma-1)}(\gamma V)^{-1/\eta}\right\} < \bar{z}$

Substitutability: $\mathcal{S}^{GTL}(V) = 1 + (\sigma-1)(\gamma V)^{1/\eta}$

Love-for-Variety: $\mathcal{L}^{GTL}(V) = \frac{1}{\sigma-1}\frac{\eta}{1+\eta}(\gamma V)^{-1/\eta} = \frac{\eta}{1+\eta}\frac{1}{\mathcal{S}^{GTL}(V)-1} < \frac{1}{\mathcal{S}^{GTL}(V)-1}$.

Gains from Trade: $\ln GT = \mathcal{L}^{GTL}(V)\frac{1-(\lambda)^{1/\eta}}{1/\eta} = \mathcal{L}^{GTL}(V^*)\left(\frac{\lambda}{1-\lambda}\right)^{-1/\eta}\frac{1-(\lambda)^{1/\eta}}{1/\eta} > 0.$

CoPaTh: $s(z) = \gamma\sigma^{\frac{\rho}{1-\rho}}\left[1-\left(\frac{z}{\bar{z}}\right)^{\frac{1-\rho}{\rho}}\right]^{\frac{\rho}{1-\rho}}$; $\bar{z} \equiv \left(1-\frac{1}{\sigma}\right)^{-\frac{\rho}{1-\rho}}$; $\rho < 1 \Rightarrow z = s^{-1}\left(\frac{1}{V}\right) = \bar{z}\left\{1-\frac{1}{\sigma}\left[\frac{1}{\gamma V}\right]^{\frac{1-\rho}{\rho}}\right\}^{\frac{\rho}{1-\rho}} < \bar{z},$

Substitutability: $\mathcal{S}^{CPT}(V) = \zeta^{s}\left(s^{-1}\left(\frac{1}{V}\right)\right) = \sigma[\gamma V]^{\frac{1-\rho}{\rho}} > 1.$

Love-for-Variety: $\mathcal{L}^{CPT}(V) = \sum_{n=0}^{\infty}\frac{\rho}{1+(1-\rho)n}\left[\frac{1}{\mathcal{S}^{CPT}(V)}\right]^{n+1} = \sum_{n=0}^{\infty}\frac{\rho}{1+(1-\rho)n}\left[\frac{1}{\sigma(\gamma V)^{\frac{1-\rho}{\rho}}}\right]^{n+1}.$

Gains from Trade: $\ln GT = -\sum_{n=0}^{\infty}\frac{\rho}{1+(1-\rho)n}\frac{1-(\lambda)^{\frac{1-\rho}{\rho}(n+1)}}{\sigma(\gamma V)^{\frac{1-\rho}{\rho}(n+1)}} + \ln\left[1+\frac{1-(\lambda)^{\frac{1-\rho}{\rho}}}{\sigma(\gamma V)^{\frac{1-\rho}{\rho}}-1}\right]^{\frac{\rho}{1-\rho}}.$

Homothetic Direct Implicit Additivity (HDIA)

Symmetric HDIA (Homothetic Directly Implicitly Additive) DS with Gross Substitutes

Definition: A symmetric CRS technology, $X = X(\mathbf{x}) \equiv Z\hat{X}(\mathbf{x})$ is called *homothetic with direct implicit additivity* (HDIA) with gross substitutes if it can be defined implicitly by:

$$\int_{\Omega} \phi\left(\frac{Zx_\omega}{X(\mathbf{x})}\right) d\omega = \int_{\Omega} \phi\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right) d\omega \equiv 1,$$

where $\phi(\cdot): \mathbb{R}_+ \to \mathbb{R}_+$ is independent of $Z > 0$, C^3, with $\phi(0) = 0; \phi(\infty) = \infty; \phi'(\infty) = 0, \phi'(y) > 0 > \phi''(y)$, $-y\phi''(y)/\phi'(y) < 1, \forall y \in (0, \infty)$.

- By construction, $\hat{X}(\mathbf{x})$ is independent of $Z > 0$, TFP.
- If $\phi'(0) < \infty$, the choke price is $B(\mathbf{p})\phi'(0)$. If $\phi'(0) = \infty$, no choke price.
- CES with gross substitutes: $\phi(y) = (y)^{1-1/\sigma}$, $(\sigma > 1)$.
- CoPaTh: $\phi(y) = \int_0^{y}\left(1 + \frac{1}{\sigma-1}(\xi)^{\frac{1-\rho}{\rho}}\right)^{\frac{\rho}{\rho-1}} d\xi$, $0 < \rho < 1$, converging to CES with $\rho \nearrow 1$.
- An extension of the Kimball (1995) aggregator in the sense that Ω is not fixed and $V \equiv |\Omega|$ is a variable.

Inverse Demand Curve:	$\frac{p_\omega}{B(\mathbf{p})} = \phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right) = \phi'\left(\frac{Zx_\omega}{X(\mathbf{x})}\right)$	**Demand Curve:**	$\frac{Zx_\omega}{X(\mathbf{x})} = \frac{x_\omega}{\hat{X}(\mathbf{x})} = (\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)$
Unit Cost Function:	$P(\mathbf{p}) = \frac{1}{Z}\hat{P}(\mathbf{p}) \equiv \frac{1}{Z}\int_{\Omega} p_\omega (\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right) d\omega$		

where $B(\mathbf{p})$ and $\hat{P}(\mathbf{p})$ are both independent of $Z > 0$ and

$$\int_{\Omega} \phi\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right) d\omega \equiv 1.$$

Budget Share:	$s_\omega \equiv \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{p_\omega}{\hat{P}(\mathbf{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right) = \frac{x_\omega}{C^*(\boldsymbol{x})}\phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right),$

where

$$C^*(\mathbf{x}) \equiv \int_\Omega x_\omega \phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right) d\omega$$

satisfying the identity

$$\frac{\hat{P}(\mathbf{p})}{B(\mathbf{p})} = \int_\Omega \frac{p_\omega}{B(\mathbf{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right) d\omega = \int_\Omega \phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right)\frac{x_\omega}{\hat{X}(\mathbf{x})} d\omega = \frac{C^*(\mathbf{x})}{\hat{X}(\mathbf{x})}.$$

Budget Share under HDIA: A function of the two relative prices, $p_\omega/\hat{P}(\mathbf{p})$ & $p_\omega/B(\mathbf{p})$, or of the two relative quantities, $x_\omega/\hat{X}(\mathbf{x})$ & $x_\omega/C^*(\mathbf{x})$, unless $\hat{P}(\mathbf{p})/B(\mathbf{p}) = C^*(\mathbf{x})/\hat{X}(\mathbf{x})$ is a constant, which occurs iff CES.

Price Elasticity:	$\zeta_\omega = \zeta^*(x_\omega;\mathbf{x}) = -\frac{\phi'(y_\omega)}{y_\omega \phi''(y_\omega)} \equiv \zeta^D(y_\omega) = \zeta^D\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right) = \zeta(p_\omega;\mathbf{p}) > 1$

Notes:

- Price Elasticity, unlike the budget share, is a function of a single variable, $y_\omega \equiv x_\omega/\hat{X}(\mathbf{x})$ or $\phi'(y_\omega) = p_\omega/B(\mathbf{p})$.
- $\zeta^D(y_\omega) = \sigma > 1$ under CES, $\phi(y) = (y)^{1-1/\sigma}$
- The 2nd law iff $\zeta^{D'}(\cdot) < 0$, satisfied by $\zeta^D(y) = 1 + (\sigma - 1)(y)^{\frac{\rho-1}{\rho}}$ under CoPaTh.

For symmetric quantity patterns, $\mathbf{x} = x\mathbf{1}_\Omega$,

$$\phi\left(\frac{1}{\hat{X}(\mathbf{1}_\Omega)}\right)V = 1 \implies \frac{1}{\hat{X}(\mathbf{1}_\Omega)} = \phi^{-1}\left(\frac{1}{V}\right); \quad \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 = \frac{d\ln \phi^{-1}(1/V)}{d\ln(1/V)} - 1.$$

Hence,

	HDIA
Substitutability	$\mathcal{S}^D(V) \equiv \zeta^D\left(\phi^{-1}\left(\frac{1}{V}\right)\right) > 1, \quad where\ \zeta^D(y) \equiv -\frac{\phi'(y)}{y\phi''(y)} > 1.$
Love-for-variety	$\mathcal{L}^D(V) \equiv \frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/V))} - 1 > 0, \quad where\ 0 < \mathcal{E}_\phi(y) \equiv \frac{y\phi'(y)}{\phi(y)} < 1.$

Note:

Since $\phi^{-1}(1/V)$ is decreasing in V,

$$\zeta^{D\prime}(\cdot) \lesseqgtr 0 \Leftrightarrow \mathcal{S}^{D\prime}(\cdot) \gtreqless 0; \ \ \mathcal{E}'_\phi(\cdot) \gtreqless 0 \Leftrightarrow \mathcal{L}^{D\prime}(\cdot) \lesseqgtr 0.$$

Furthermore,

$$\mathcal{E}_\phi(y) \equiv \frac{y\phi'(y)}{\phi(y)} = \frac{\int_0^y d[\xi\phi'(\xi)]}{\phi(y)} = \frac{\int_0^y[\xi\phi''(\xi) + \phi'(\xi)]d\xi}{\phi(y)} = \int_0^y \left[1 - \frac{1}{\zeta^D(\xi)}\right] w^D(\xi; y)d\xi$$

where $w^D(\xi; y) \equiv \phi'(\xi)/\phi(y) > 0$, which satisfies $\int_0^y w^D(\xi;\ y)\, d\xi = 1$. Hence,

$$\frac{y\mathcal{E}'_\phi(y)}{\mathcal{E}_\phi(y)} = 1 - \frac{1}{\zeta^D(y)} - \mathcal{E}_\phi(y) = \int_0^y \left[\frac{1}{\zeta^D(\xi)}\right] w^D(\xi; y)d\xi - \frac{1}{\zeta^D(y)}.$$

Proposition D-1:

$$\frac{\mathcal{y}\mathcal{E}_\phi'(\mathcal{y})}{\mathcal{E}_\phi(\mathcal{y})} = 1 - \frac{1}{\zeta^D(\mathcal{y})} - \mathcal{E}_\phi(\mathcal{y}) = \int_0^{\mathcal{y}} \left[\frac{1}{\zeta^D(\xi)}\right] w^D(\xi;\mathcal{y})d\xi - \frac{1}{\zeta^D(\mathcal{y})}.$$

where $w^D(\xi;\mathcal{y}) \equiv \phi'(\xi)/\phi(\mathcal{y}) > 0$, which satisfies $\int_0^{\mathcal{y}} w^D(\xi;\, \mathcal{y})\, d\xi = 1.$ Hence,

$$\zeta^{D'}(\mathcal{y}) \lesseqgtr 0, \forall \mathcal{y} \in (0, \mathcal{y}_0) \implies \mathcal{E}_\phi'(\mathcal{y}) \lesseqgtr 0, \forall \mathcal{y} \in (0, \mathcal{y}_0).$$

The opposite is not true in general. However,

$$\zeta^{D'}(\mathcal{y}) = 0, \forall \mathcal{y} \in (0, \mathcal{y}_0) \iff \mathcal{E}_\phi'(\mathcal{y}) = 0, \forall \mathcal{y} \in (0, \mathcal{y}_0).$$

Proposition D-2: For $\phi(\mathcal{y}_0)V_0 = 1$,

$$\zeta^{D'}(\mathcal{y}) \lesseqgtr 0\ \forall \mathcal{y} \in (0, \mathcal{y}_0) \Leftrightarrow \mathcal{S}^{D\prime}(V) \gtreqless 0, \forall V \in (V_0, \infty);$$

$$\mathcal{E}_\phi'(\mathcal{y}) \lesseqgtr 0, \forall \mathcal{y} \in (0, \mathcal{y}_0) \Leftrightarrow \mathcal{L}^{D\prime}(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

Moreover,

$$\mathcal{S}^{D\prime}(V) \gtreqless 0, \forall V \in (V_0, \infty) \implies \mathcal{L}^{D\prime}(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

The opposite is not true in general. However,

$$\mathcal{S}^{D\prime}(V) = 0, \forall V \in (V_0, \infty) \iff \mathcal{L}^{D\prime}(V) = 0, \forall V \in (V_0, \infty).$$

Proposition D-3: $\mathcal{L}'(V) \lesseqgtr 0 \Leftrightarrow \frac{\mathcal{y}\mathcal{E}_\phi'(\mathcal{y})}{\mathcal{E}_\phi(\mathcal{y})} = 1 - \frac{1}{\zeta^D(\mathcal{y})} - \mathcal{E}_\phi(\mathcal{y}) \lesseqgtr 0 \Leftrightarrow \mathcal{L}(V) \lesseqgtr \frac{1}{\mathcal{S}(V)-1}.$

Proposition D-4.

$$\lim_{V\to\infty} \mathcal{L}(V) = \lim_{\mathcal{y}\to 0} \frac{\phi(\mathcal{y})}{\mathcal{y}\phi'(\mathcal{y})} - 1 = \lim_{\mathcal{y}\to 0} \frac{\phi'(\mathcal{y})}{\phi'(\mathcal{y}) + \mathcal{y}\phi''(\mathcal{y})} - 1 = \lim_{\mathcal{y}\to 0} \frac{1}{\zeta^D(\mathcal{y}) - 1} = \lim_{V\to\infty} \frac{1}{\mathcal{S}(V) - 1}.$$

Gains from Trade under HDIA

$$\ln(GT) = \int_{V}^{V+V^*} \mathcal{L}^D(v)\frac{dv}{v} = \int_{V}^{V/\lambda}\left[\frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/v))} - 1\right]\frac{dv}{v}.$$

Using the change of variables, $\phi^{-1}\left(\frac{1}{v}\right) = \xi \Leftrightarrow v = \frac{1}{\phi(\xi)}$, which implies $\frac{dv}{v} = -\frac{\xi\phi'(\xi)}{\phi(\xi)}\frac{d\xi}{\xi} = -\mathcal{E}_\phi(\xi)\frac{d\xi}{\xi}$,

$$\ln(GT) = \int_{V}^{V/\lambda}\left[\frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/v))} - 1\right]\frac{dv}{v} = \int_{\phi^{-1}(1/V)}^{\phi^{-1}(\lambda/V)}\left[\mathcal{E}_\phi(\xi) - 1\right]\frac{d\xi}{\xi}$$

$$= -\int_{\phi^{-1}(1/V)}^{\phi^{-1}(\lambda/V)}\frac{d\xi}{\xi} + \int_{\phi^{-1}(1/V)}^{\phi^{-1}(\lambda/V)}\left[\mathcal{E}_\phi(\xi)\right]\frac{d\xi}{\xi} = \ln\left[\frac{(\lambda/V)}{\phi^{-1}(\lambda/V)}\frac{\phi^{-1}(1/V)}{(1/V)}\right].$$

Hence,

$$GT = \frac{(\lambda/V)}{\phi^{-1}(\lambda/V)}\frac{\phi^{-1}(1/V)}{(1/V)} = \frac{((1-\lambda)/V^*)}{\phi^{-1}((1-\lambda)/V^*)}\frac{\phi^{-1}((1-\lambda)/\lambda V^*)}{((1-\lambda)/\lambda V^*)}.$$

Homothetic Indirect Implicit Additivity (HIIA)

Symmetric HIIA (Homothetic Indirectly Implicitly Additive) DS with Gross Substitutes

Definition: A symmetric CRS technology, $P = P(\mathbf{p}) = \hat{P}(\mathbf{p})/Z$, is called *homothetic with indirect implicit additivity* (HIIA) if it can be defined implicitly by:

$$\int_{\Omega} \theta\left(\frac{p_\omega}{ZP(\mathbf{p})}\right) d\omega = \int_{\Omega} \theta\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right) d\omega = 1,$$

where $\theta: \mathbb{R}_{++} \to \mathbb{R}_{+}$ is independent of $Z > 0, C^3$, with $\theta(z) > 0, \theta'(z) < 0, \theta''(z) > 0, -z\theta''(z)/\theta'(z) > 1$, for $\theta(z) > 0$ with $\lim_{z\to 0} \theta(z) = \infty$ and $\lim_{z\to \bar{z}} \theta(z) = \lim_{z\to \bar{z}} \theta'(z) = 0$, where $\bar{z} \equiv \inf\{z > 0 | \theta(z) = 0\}$.

- By construction, $\hat{P}(\mathbf{p})$ is independent of $Z > 0$, TFP.
- If $\bar{z} < \infty$, $\hat{P}(\mathbf{p})\bar{z}$ is the choke price. If $\bar{z} = \infty$, no choke price.
- CES with gross substitutes: $\theta(z) = (z)^{1-\sigma}$, $(\sigma > 1)$.
- CoPaTh: $\theta(z) = \sigma^{\frac{\rho}{1-\rho}} \int_{z/\bar{z}}^{1} \left((\xi)^{\frac{\rho-1}{\rho}} - 1\right)^{\frac{\rho}{1-\rho}} d\xi$ for $z < \bar{z} = \left(\frac{\sigma}{\sigma-1}\right)^{\frac{\rho}{1-\rho}}$; $0 < \rho < 1$, converging to CES as $\rho \nearrow 1$.

Inverse Demand Curve:	$\frac{p_\omega}{ZP(\mathbf{p})} = \frac{p_\omega}{\hat{P}(\mathbf{p})} = (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)$	**Demand Curve:**	$\frac{x_\omega}{B^*(\mathbf{x})} = -\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right) = -\theta'\left(\frac{p_\omega}{ZP(\mathbf{p})}\right) > 0$
Production Function:	$X(\mathbf{x}) = Z\hat{X}(\mathbf{x}) \equiv Z \int_{\Omega} (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right) x_\omega d\omega$		

where $\hat{X}(\mathbf{x})$ and $B^*(\mathbf{x})$ are both independent of $Z > 0$ and

$$\int_{\Omega} \theta\left((-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\right) d\omega \equiv 1.$$

Budget Share:	$$\frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\boldsymbol{x})}\right)\frac{x_\omega}{\hat{X}(\mathbf{x})} = -\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)\frac{p_\omega}{C(\mathbf{p})}$$

where

$$C(\mathbf{p}) \equiv -\int_\Omega \theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)p_\omega d\omega > 0$$

satisfying the identity,

$$\frac{C(\mathbf{p})}{\hat{P}(\mathbf{p})} = \int_\Omega \frac{p_\omega}{\hat{P}(\mathbf{p})}\left[-\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)\right]d\omega = \int_\Omega (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\frac{x_\omega}{B^*(\mathbf{x})}d\omega = \frac{\hat{X}(\mathbf{x})}{B^*(\mathbf{x})}.$$

Budget share under HIIA: A function of two relative prices, $p_\omega/\hat{P}(\mathbf{p})$ and $p_\omega/C(\mathbf{p})$, or of two relative quantities, $x_\omega/\hat{X}(\mathbf{x})$ and $x_\omega/B^*(\mathbf{x})$, unless $C(\mathbf{p})/\hat{P}(\mathbf{p}) = \hat{X}(\mathbf{x})/B^*(\mathbf{x})$ is a constant, which occurs iff CES.

Price Elasticity:	$$\zeta_\omega = \zeta(p_\omega;\mathbf{p}) = -\frac{z_\omega\theta''(z_\omega)}{\theta'(z_\omega)} \equiv \zeta^I(z_\omega) = \zeta^I\left((-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\right) = \zeta^*(x_\omega;\ \mathbf{x}) > 1$$

Notes:

- Price Elasticity, unlike the budget share, is a function of a single variable, $z_\omega \equiv p_\omega/\hat{P}(\mathbf{p})$ or $x_\omega/B^*(\mathbf{x}) = -\theta'(z_\omega)$.
- $\zeta^I(z_\omega) = \sigma > 1$ under CES, $\theta(z) = (z)^{1-\sigma}$, $(\sigma > 1)$.
- The 2nd law iff $\zeta^{I\prime}(z_\omega) > 0$, satisfied by $\zeta^I(z_\omega) = \frac{\sigma}{\sigma-(\sigma-1)(z_\omega)^{(1-\rho)/\rho}} = \frac{1}{1-(z_\omega/\bar{z})^{(1-\rho)/\rho}}$ under CoPaTh.

For symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$,

$$\theta\left(\frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})}\right)V = 1 \implies \frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})} = \theta^{-1}\left(\frac{1}{V}\right); \qquad -\frac{d\ln P(\mathbf{1}_\Omega)}{d\ln V} = -\frac{d\ln\theta^{-1}(1/V)}{d\ln(1/V)}$$

Hence,

	HIIA
Substitutability	$\mathcal{S}^I(V) \equiv \zeta^I\left(\theta^{-1}\left(\frac{1}{V}\right)\right), \quad where\ \zeta^I(z) \equiv -\frac{z\theta''(z)}{\theta'(z)} > 1$
Love-for-variety	$\mathcal{L}^I(V) \equiv \frac{1}{\mathcal{E}_\theta\left(\theta^{-1}(1/V)\right)}, \quad where\ \mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} > 0.$

Since $\theta^{-1}(1/V)$ is increasing in V,

$$\zeta^{I\prime}(\cdot) \gtreqless 0 \Leftrightarrow \mathcal{S}^{I\prime}(\cdot) \gtreqless 0; \qquad \mathcal{E}_\theta'(\cdot) \lesseqgtr 0 \Leftrightarrow \mathcal{L}^{I\prime}(\cdot) \gtreqless 0.$$

Furthermore,

$$\mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} = \frac{\bar{z}\theta'(\bar{z}) - z\theta'(z)}{\theta(z)} = \frac{\int_z^{\bar{z}} d[\xi\theta'(\xi)]}{\theta(z)} = \frac{\int_z^{\bar{z}}[\theta'(\xi) + \xi\theta''(\xi)]d\xi}{\theta(z)} = \int_z^{\bar{z}} [\zeta^I(\xi) - 1]w^I(\xi; z)d\xi,$$

where $w^I(\xi; z) \equiv -\theta'(\xi)/\theta(z) > 0$, which satisfies $\int_z^{\bar{z}} w^I(\xi; z)\, d\xi = 1$. Hence,

$$\frac{z\mathcal{E}_\theta'(z)}{\mathcal{E}_\theta(z)} = \mathcal{E}_\theta(z) + 1 - \zeta^I(z) = \int_z^{\bar{z}} \zeta^I(\xi)w^I(\xi; z)d\xi - \zeta^I(z).$$

From this,

Proposition I-1:

$$\frac{z\mathcal{E}_\theta'(z)}{\mathcal{E}_\theta(z)} = \mathcal{E}_\theta(z) + 1 - \zeta^I(z) = \int_z^{\overline{z}} \zeta^I(\xi) w^I(\xi; z) d\xi - \zeta^I(z).$$

where $w^I(\xi; z) \equiv -\theta'(\xi)/\theta(z) > 0$, which satisfies $\int_z^{\overline{z}} w^I(\xi; z)\, d\xi = 1$.

$$\zeta^{I'}(z) \gtreqless 0, \forall z \in (z_0, \overline{z}) \quad \Longrightarrow \quad \mathcal{E}_\theta'(z) \gtreqless 0, \forall z \in (z_0, \overline{z}).$$

The opposite is not true in general. However,

$$\zeta^{I'}(z) = 0, \forall z \in (z_0, \overline{z}) \quad \Leftrightarrow \quad \mathcal{E}_\theta'(z) = 0, \forall z \in (z_0, \overline{z}).$$

Proposition I-2: For $\theta(z_0)V_0 = 1$,

$$\zeta^{I'}(z) \gtreqless 0, \forall z \in (z_0, \overline{z}) \Leftrightarrow \mathcal{S}^{I\prime}(V) \gtreqless 0, \forall V \in (V_0, \infty);$$

$$\mathcal{E}_\theta'(z) \gtreqless 0, \forall z \in (z_0, \overline{z}) \Leftrightarrow \mathcal{L}^{I\prime}(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

Moreover,

$$\mathcal{S}^{I\prime}(V) \gtreqless 0, \forall V \in (V_0, \infty) \Longrightarrow \mathcal{L}^{I\prime}(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

The opposite is not true in general. However,

$$\mathcal{S}^{I\prime}(V) = 0, \forall V \in (V_0, \infty) \Leftrightarrow \mathcal{L}^{I\prime}(V) = 0, \forall V \in (V_0, \infty).$$

Proposition I-3: $\mathcal{L}^{I\prime}(V) \lesseqgtr 0 \Leftrightarrow \frac{z\mathcal{E}_\theta'(z)}{\mathcal{E}_\theta(z)} = \mathcal{E}_\theta(z) + 1 - \zeta^I(z) \gtreqless 0 \Leftrightarrow \mathcal{L}^I(V) \lesseqgtr \frac{1}{\mathcal{S}^I(V) - 1}$.

Proposition I-4.

$$\lim_{V\to\infty} \mathcal{L}^I(V) = -\lim_{z\to\overline{z}} \frac{\theta(z)}{z\theta'(z)} = -\lim_{z\to\overline{z}} \frac{\theta'(z)}{\theta'(z) + z\theta''(z)} = \lim_{z\to\overline{z}} \frac{1}{\zeta^I(z) - 1} = 0.$$

Gains from Trade under HIIA:

$$\ln(GT) = \int_{V}^{V+V^*} \mathcal{L}^I(v)\frac{dv}{v} = \int_{V}^{V/\lambda} \frac{1}{\mathcal{E}_\theta\left(\theta^{-1}(1/v)\right)}\frac{dv}{v}.$$

Using the change of variables, $\theta^{-1}\left(\frac{1}{v}\right) = \xi \Leftrightarrow v = \frac{1}{\theta(\xi)}$, which implies $\frac{dv}{v} = -\frac{\xi\theta'(\xi)}{\theta(\xi)}\frac{d\xi}{\xi} = \mathcal{E}_\theta(\xi)\frac{d\xi}{\xi}$,

$$\ln(GT) = \int_{V}^{V/\lambda} \frac{1}{\mathcal{E}_\theta\left(\theta^{-1}(1/v)\right)}\frac{dv}{v} = \int_{\theta^{-1}(1/V)}^{\theta^{-1}(\lambda/V)} \frac{d\xi}{\xi} = \ln\left(\frac{\theta^{-1}(\lambda/V)}{\theta^{-1}(1/V)}\right).$$

Hence,

$$GT = \frac{\theta^{-1}(\lambda/V)}{\theta^{-1}(1/V)} = \frac{\theta^{-1}((1-\lambda)/V^*)}{\theta^{-1}((1-\lambda)/\lambda V^*)}.$$

Appendix C: An Alternative (and Equivalent) Definition of H.S.A.

Definition: A symmetric CRS technology, X= $X(\mathbf{x})$ is called *homothetic single aggregator* (H.S.A.) if the budget share of ω depends solely on a single variable, $y_\omega \equiv x_\omega/A^*$, its own quantity x_ω, normalized by the common quantity aggregator, $A^* = A^*(\mathbf{x})$. $$s_\omega \equiv \frac{p_\omega x_\omega}{\mathbf{px}} = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} = s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right), \qquad \text{where} \quad \int_\Omega s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) d\omega \equiv 1.$$

- $s^*: \mathbb{R}_{++} \to \mathbb{R}_+$: the *budget share function* in $y_\omega \equiv x_\omega/A^*$ with $0 < \mathcal{E}_{s^*}(y) \equiv \frac{d \ln s^*(y)}{d \ln y} < 1$, $s^*(0) = 0$, $s^*(\infty) = \infty$.
- $A^* = A^*(\mathbf{x})$: **the common quantity aggregator**, defined by **the adding-up constraint**, $\int_\Omega s^*(x_\omega/A^*)d\omega \equiv 1$.
  By construction, the budget shares add up to one. $A^*(\mathbf{x})$ linear homogenous in $\mathbf{x}$ for a fixed Ω. A larger Ω raiss A^*.

Price Elasticity:	$$\zeta_\omega = \zeta^*(x_\omega; \mathbf{x}) = \left[1 - \frac{d \ln s^*(y_\omega)}{d \ln y_\omega}\right]^{-1} \equiv \zeta^{S*}(y_\omega) > 1,$$

Notes: Also a function of a single variable, $y_\omega \equiv x_\omega/A^*(\mathbf{x})$.

- $\zeta^{S*}(y) = \sigma > 1$ under CES, $s^*(y) = \gamma^{1/\sigma}(y)^{1-1/\sigma}$.
- The 2nd law, $\partial\zeta(x_\omega; \mathbf{x})/\partial x_\omega < 0$, holds iff $\zeta^{S*'}(\cdot) < 0$.
- The choke price exists iff $\lim_{y\to 0} s^{*'}(y) = \bar{z} < \infty$, which implies $\lim_{y\to 0} \frac{d \ln s^*(y)}{d \ln y} = 1$ and $\lim_{y\to 0} \zeta^{S*}(y) = \infty$. E.g.,
  - Translog corresponds to $s^*(y)$, defined implicitly by $s^* \exp(s^*/\gamma) \equiv \bar{z}y$, for $\bar{z} < \infty$.
  - CoPaTh corresponds to $s^*(y) = \left[\frac{1}{\sigma} + \left(1 - \frac{1}{\sigma}\right) y^{-\frac{1-\rho}{\rho}}\right]^{-\frac{\rho}{1-\rho}} = \left[1 - \bar{z}^{-\frac{1-\rho}{\rho}} + (y\bar{z})^{-\frac{1-\rho}{\rho}}\right]^{-\frac{\rho}{1-\rho}}$.

Production Function: By integrating $\frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} = s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right)$,

$$\ln\left[\frac{X(\mathbf{x})}{c^* A^*(\mathbf{x})}\right] = \int_\Omega \left[\int_0^{\frac{x_\omega}{A^*(\mathbf{x})}} \frac{s^*(\xi^*)}{\xi^*} d\xi^*\right] d\omega = \int_\Omega s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) \Phi^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) d\omega = \int_\Omega H^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) d\omega$$

where $c^* > 0$ is a constant, proportional to TFP, and

$$H^*(y) \equiv \int_0^y \frac{s^*(\xi^*)}{\xi^*} d\xi^* > 0 \Rightarrow H^{*'}(y) = \frac{s^*(y)}{y} > 0 \Rightarrow \frac{yH^{*''}(y)}{H^{*'}(y)} = \frac{ys^{*'}(y)}{s^*(y)} - 1 < 0.$$

$$\Phi^*(y) \equiv \frac{1}{s^*(y)} \int_0^y \frac{s^*(\xi^*)}{\xi^*} d\xi^* \equiv \frac{H^*(y)}{s^*(y)} = \frac{H^*(y)}{yH^{*'}(y)} > 1,$$

$$\zeta^{S*}(y) \equiv \left[1 - \frac{ys^{*'}(y)}{s^*(y)}\right]^{-1} \equiv -\frac{H^{*'}(y)}{yH^{*''}(y)} > 1$$

Notes:

- $X(\mathbf{x})$, linear homogeneous, monotonic, and strictly quasi-concave, ensuring the integrability of H.S.A.
- $X(\mathbf{x})/A^*(\mathbf{x})$is not constant and depends on $\mathbf{x}$, with the sole exception of CES, because

$$\frac{\partial \ln A^*(\mathbf{x})}{\partial \ln x_\omega} = \frac{y_\omega {s^*}'(y_\omega)}{\int_\Omega {s^*}'(y_{\omega'}) y_{\omega'} d\omega'} = \frac{\left[1 - \frac{1}{\zeta^{S*}(y_\omega)}\right] s^*(y_\omega)}{\int_\Omega \left[1 - \frac{1}{\zeta^{S*}(y_{\omega'})}\right] s^*(y_{\omega'}) d\omega'} \neq \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} = s^*(y_\omega),$$

unless $\zeta^{S*}(y) = \sigma > 1 \Leftrightarrow s^*(y) = \gamma^{1/\sigma}(y)^{1-1/\sigma} \Leftrightarrow \Phi^*(y) = \sigma/(\sigma - 1) \Leftrightarrow H^*(y) = \gamma^{1/\sigma}(y)^{1-1/\sigma}\, \sigma/(\sigma - 1)$.

For symmetric quantity patterns, $\mathbf{x} = x\mathbf{1}_\Omega$,

$$1 = s^*\left(\frac{x}{A^*(\mathbf{x})}\right)V = s^*\left(\frac{1}{A^*(\mathbf{1}_\Omega)}\right)V \Rightarrow \frac{1}{A^*(\mathbf{1}_\Omega)} = s^{*-1}\left(\frac{1}{V}\right) \Rightarrow \mathcal{S}(V) = \zeta^{S*}\left(s^{*-1}\left(\frac{1}{V}\right)\right)$$

$$\ln\frac{X(\mathbf{1}_\Omega)}{c^*} = \Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right) - \ln s^{*-1}\left(\frac{1}{V}\right) \Rightarrow$$

$$\mathcal{L}(V) + 1 \equiv \frac{d \ln X(\mathbf{1}_\Omega)}{d \ln V} = \left.\frac{d[\ln y - \Phi^*(y)]}{d \ln s^*(y)}\right|_{y=s^{*-1}\left(\frac{1}{V}\right)} = \left.\left[\frac{d[\ln y - \Phi^*(y)]}{d \ln y} \Big/ \frac{d \ln s^*(y)}{d \ln y}\right]\right|_{z=s^{*-1}\left(\frac{1}{V}\right)} = \Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right).$$

Hence,

	H.S.A.
Substitutability	$\mathcal{S}(V)=\zeta^{S*}\left(s^{*-1}\left(\frac{1}{V}\right)\right)$, where $\zeta^{S*}(y)\equiv\left[1-\frac{ys^{*'}(y)}{s^*(y)}\right]^{-1}\equiv-\frac{H^{*'}(y)}{yH^{*''}(y)}>1.$
Love-for-variety	$\mathcal{L}(V)=\Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right)-1>0$, where $\Phi^*(y)\equiv\frac{1}{s^*(y)}\int_0^y\frac{s^*(\xi^*)}{\xi^*}d\xi^*=\frac{H^*(y)}{yH^{*'}(y)}>1,$

Notes:

Since $s^{*-1}(1/V)$ is decreasing in V,

$$\zeta^{S*'}(\cdot)\lesseqgtr 0\Leftrightarrow\mathcal{S}'(\cdot)\gtreqless 0.$$
$$\Phi^{*'}(\cdot)\lesseqgtr 0\Leftrightarrow\mathcal{L}'(\cdot)\gtreqless 0.$$

Furthermore,

$$\frac{1}{\Phi^*(y)}=\frac{yH^{*'}(y)}{H^*(y)}=\frac{\int_0^y d[\xi^*H^{*'}(\xi^*)]}{H^*(y)}=\frac{\int_0^y[\xi^*H^{*''}(\xi^*)+H^{*'}(\xi)]d\xi^*}{H^*(y)}=\int_0^y\left[1-\frac{1}{\zeta^{S*}(\xi^*)}\right]w^{S*}(\xi^*;y)d\xi^*,$$

where $w^{S*}(\xi^*;y)\equiv H^{*'}(\xi^*)/H^*(y)>0$, which satisfies $\int_0^y w^{S*}(\xi^*;y)d\xi^*=1$. Hence,

$$\frac{y\Phi^{*'}(y)}{\Phi^*(y)}=\frac{yH^{*'}(y)}{H^*(y)}-1-\frac{yH^{*''}(y)}{H^{*'}(y)}=\frac{1}{\Phi^*(y)}-1+\frac{1}{\zeta^{S*}(y)}=\frac{1}{\zeta^{S*}(y)}-\int_0^y\left[\frac{1}{\zeta^{S*}(\xi^*)}\right]w^{S*}(\xi^*;y)d\xi^*.$$

Proposition S*-1

$$\frac{y\Phi^{*\prime}(y)}{\Phi^*(y)} = \frac{1}{\Phi^*(y)} - 1 + \frac{1}{\zeta^{S*}(y)} = \frac{1}{\zeta^{S*}(y)} - \int_0^y \left[\frac{1}{\zeta^{S*}(\xi^*)}\right] w^*(\xi^*; y) d\xi^*.$$

where $w^{S*}(\xi^*; y) \equiv H^{*\prime}(\xi^*)/H^*(y) > 0$, which satisfies $\int_0^y w^*(\xi^*; y) d\xi^* = 1$. Hence,

$$\zeta^{S*\prime}(y) \lesseqgtr 0, \forall y \in (0, y_0) \Longrightarrow \Phi^{*'}(y) \gtreqless 0, \forall y \in (0, y_0).$$

The opposite is not true in general. However,

$$\zeta^{S*\prime}(y) = 0, \forall y \in (0, y_0) \Leftrightarrow \Phi^{*'}(y) = 0, \forall y \in (0, y_0).$$

Proposition S*-2: For $s^*(y_0)V_0 = 1$,

$$\zeta^{S*\prime}(y) \lesseqgtr 0, \forall y \in (0, y_0) \Leftrightarrow \mathcal{S}'(V) \gtreqless 0, \forall V \in (V_0, \infty);$$

$$\Phi^{*\prime}(y) \gtreqless 0, \forall y \in (0, y_0) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

Moreover,

$$\mathcal{S}'(V) \gtreqless 0, \forall V \in (V_0, \infty) \Longrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

The opposite is not true in general. However,

$$\mathcal{S}'(V) = 0, \forall V \in (V_0, \infty) \Leftrightarrow \mathcal{L}'(V) = 0, \forall V \in (V_0, \infty).$$

Proposition S*-3: $\mathcal{L}'(V) \lesseqgtr 0 \Leftrightarrow \frac{1}{\Phi^*(y)} - 1 + \frac{1}{\zeta^{S*}(y)} \gtreqless 0 \Leftrightarrow \mathcal{L}(V) \lesseqgtr \frac{1}{\mathcal{S}(V)-1}$.

Proposition S*-4.

$$\lim_{V\to\infty} \mathcal{L}(V) = \lim_{V\to\infty} \Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right) - 1 = \lim_{y\to 0} \frac{s^*(y)}{ys^{*'}(y)} - 1 = \lim_{y\to 0} \frac{\zeta^*(y)}{\zeta^*(y) - 1} - 1 = \lim_{V\to\infty} \frac{1}{\mathcal{S}(V) - 1}.$$

Equivalence of the Two Definitions of H.S.A.

Under the isomorphism given by the one-to-one mapping btw $s(z) \leftrightarrow s^*(y)$, defined by:

$$s^*(y) = s\left(\frac{s^*(y)}{y}\right); \qquad s(z) = s^*\left(\frac{s(z)}{z}\right).$$

From this,

$$\zeta^{S*}(y) \equiv \left[1 - \frac{d \ln s^*(y)}{d \ln y}\right]^{-1} = \zeta^S(z) \equiv 1 - \frac{d \ln s(z)}{d \ln z} > 1,$$

$$0 < \mathcal{E}_{s^*}(y) \equiv \frac{d \ln s^*(y)}{d \ln y} < 1 \Leftrightarrow \mathcal{E}_s(z) \equiv \frac{d \ln s(z)}{d \ln z} < 0.$$

$y_\omega \equiv x_\omega / A^*(\mathbf{x})$, and $z_\omega \equiv p_\omega / A(\mathbf{p})$, are negatively related as

$$z_\omega = \frac{s^*(y_\omega)}{y_\omega} \Leftrightarrow y_\omega = \frac{s(z_\omega)}{z_\omega},$$

$$\frac{dy_\omega}{y_\omega} = -\zeta^S(z_\omega)\frac{dz_\omega}{z_\omega} \Leftrightarrow \frac{dz_\omega}{z_\omega} = -\frac{1}{\zeta^{S*}(y_\omega)}\frac{dy_\omega}{y_\omega}$$

and

$$\frac{z_\omega \zeta^{S\prime}(z_\omega)}{y_\omega \zeta^{S*\prime}(y_\omega)} = -\zeta^S(z_\omega) = -\zeta^{S*}(y_\omega) < 0.$$

If $\lim_{y \to 0} s^{*\prime}(y) < \infty$, $\lim_{y \to 0} \zeta^{S*}(y) = \infty$ and the (normalized) choke price is:

$$\lim_{y \to 0} \frac{s^*(y)}{y} = \lim_{y \to 0} s^{*\prime}(y) = \bar{z} \equiv \inf\{z > 0 | s(z) = 0\} < \infty$$

Moreover,

$$\frac{p_\omega x_\omega}{A(\mathbf{p})A^*(\mathbf{x})} = y_\omega z_\omega = s(z_\omega) = s^*(y_\omega) = \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})}$$

hence we have the identity,

$$c \exp\left[\int_\Omega s(z_\omega)\Phi(z_\omega)d\omega\right] = \frac{A(\mathbf{p})}{P(\mathbf{p})} = \frac{X(\mathbf{x})}{A^*(\mathbf{x})} = c^* \exp\left[\int_\Omega s^*(y_\omega)\Phi^*(y_\omega)\, d\omega\right]$$

which is a constant iff CES.

Furthermore, using

$$s(\xi) = s^*(\xi^*) = \xi\xi^* \rightarrow \frac{d\xi^*}{\xi^*} = \left[\frac{\xi s'(\xi)}{s(\xi)} - 1\right]\frac{d\xi}{\xi} \rightarrow s^*(\xi^*)\frac{d\xi^*}{\xi^*} = \left[s'(\xi) - \frac{s(\xi)}{\xi}\right]d\xi$$

$$\xi = z \leftrightarrow \xi^* = y;\ \xi = \overline{z} \leftrightarrow \xi^* = 0,$$

$$\Phi^*(y) - \Phi(z) \equiv \frac{1}{s^*(y)} \int_0^y \frac{s^*(\xi^*)}{\xi^*} d\xi^* - \frac{1}{s(z)} \int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi = \frac{1}{s(z)} \int_{\bar{z}}^{z} \left[s'(\xi) - \frac{s(\xi)}{\xi} \right] d\xi - \frac{1}{s(z)} \int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi = 1.$$

Since

$$w^S(\xi; z) \equiv -\frac{H'(\xi)}{H(z)} = \frac{s(\xi)/\xi}{s(z)\Phi(z)}; \qquad w^{S*}(\xi^*; y) \equiv \frac{H^{*'}(\xi^*)}{H^*(y)} = \frac{s^*(\xi^*)/\xi^*}{s^*(y)\Phi^*(y)},$$

this implies

$$\frac{\xi w^S(\xi; z)}{\xi^* w^{S*}(\xi^*; y)} = \frac{\Phi^*(y)}{\Phi(z)} = 1 + \frac{1}{\Phi(z)} = \frac{\Phi^*(y)}{\Phi^*(y) - 1},$$

$$\frac{c}{c^*} = \exp\left[\int_\Omega [s^*(y_\omega)\Phi^*(y_\omega) - s(z_\omega)\Phi(z_\omega)] d\omega \right] = \exp\left[\int_\Omega s(z_\omega) d\omega \right] = e.$$

and

$$\mathcal{L}(V) = \Phi\left(s^{-1}(1/V)\right) = \Phi^*\left(s^{*-1}(1/V)\right) - 1.$$